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EDITED BY

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PRINTER'S INK AND THE TEACHER

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BY T. A. A. BROADBENT

To address this Association from the Presidential chair is a great honour; but the many famous predecessors have left a new President the difficult problem of finding a topic which they have not illuminated. My choice has been determined by the uniqueness of my position, for hitherto no one has been simultaneously your President and your Editor. It is possible to edit a mathematical periodical without knowing or learning much about mathematics; but one learns a great deal about mathematicians, and a little, at any rate, about the presentation of mathematics. I had thought of calling this address "What I have learned in 25 years", but this suggestion aroused the criticisms, on the one hand, that many hours would be needed to cover the ground, and on the other, that the material would be exhausted in five minutes. To evade the sweep of this two-handed engine, the title was abandoned but the subject retained.

It would be easy, and perhaps entertaining, to fill an hour with anecdotes of the oddities which have come my way. For example, the circle-squarer flourishes very much as he did a hundred years ago in the pages of De Morgan's *Budget of Paradoxes*. He regards the problem as important, he does not know that it has been dead and buried these 80 years, he is certain not only of the accuracy of his verbose obscurities and his spidery diagrams, but also of the existence of some enormous financial reward awaiting, in some unspecified spot, the fortunate solver. One such gentleman pursued me to my home. The critical point in his argument occurred when he equated the length of a particular circular arc to that of a particular straight line in his diagrams; when asked for his grounds for this assertion, his reply was that he did not see any reason why it should not be so. He professed recognition of his error, but soon relapsed, for within a few days I had an abusive letter from him, in which he expressed his renewed faith in his own arguments, and accused me, along with my fellow teachers, of concealing the true value of π (namely, his own) from thousands of eager and innocent children, a racket, so he termed it,

from which my fellow culprits and I acquired vast sums of money. My reply on a post-card, may have been ambiguous : " How I wish you were right ".

Again, it would be possible, and, I believe, instructive, to discuss the direct and immediate relations between mathematics and typography. How have the requirements of the mathematician influenced the art of the printer? How have the flexibilities and rigidities of type-setting contributed to the development of mathematical notation? Have the two sides collaborated as fully as they might have done? Possibly not. Many American school texts on mathematics come my way, and on the whole I judge them to be somewhat inferior to our own. Apart from their content, the typography has not the polished air which only the skilled craftsman can give ; but what is impressive is the obvious freshness of ideas, the willingness to experiment with layout and variety of type-founts, the effort to exploit all the resources of the printer in order to produce a stimulating, even provocative, atmosphere. Not all such experiments are successful ; it may be doubted whether the comic-strip method has much future in mathematics. But the co-operation of author and printer is commendable. In this country too often the author never considers what his work will look like in print, or what variety of type might be effective ; he leaves all that to the printer or publisher, while sometimes, on the other hand, the printer is inclined to say that what was good enough for Todhunter is good enough for 1954. Much attention has of course been given to the printing of advanced mathematics, but very little, as far as I am aware, to author-printer problems at the school level. For instance, a hundred years ago it seems to have been accepted that the younger the child the smaller the type-size : we have now gone to the other extreme ; the younger the child the larger the type-size. Have we ever seriously sought for the optimum size as a function of age? Have we ever studied the effective diagrams which embellish some of the German school texts, and endeavoured to use their techniques to improve on the inaccurate and slovenly figures which occasionally disfigure our books? If ever the Teaching Committee were to be short of work and the Association indecently opulent, a committee on the printing of school mathematics, strengthened, as I am sure it could be, by the collaboration of experts from the publishing and printing trades, would find a splendid and largely unexplored field of fruitful work of permanent value.

Fascinating, at any rate to me, as these topics are, I wish to speak this morning of other relations between mathematics and its printed exposition, less direct, less obvious, but, I believe, no less valuable to the teacher. I have no new doctrine to urge, no novel theory to expound ; but certain matters, points of faith to good teachers for many years, have impressed me with their importance more emphatically through my experiences as an editor.

Let me begin then with a story concerning that great American mathematician and physicist, Josiah Willard Gibbs. Gibbs, it is said, was a most reticent man, and during the long period of his service on the Senate of the University of Yale, he spoke only once. The occasion was a demand from the department of modern languages for more facilities, these to be provided by a sacrifice on the part of the department of mathematics, the demand concluding with an eloquent peroration depreciating the value of mathematics in comparison with the educational and cultural status of languages. All eyes turned to the taciturn Gibbs. Eventually he rose, pronounced one brief sentence and sat down again. What he said was : " Mathematics *is* a language."

To anyone concerned with the process of embodying the abstract ideas of mathematics in the cold permanence of type, this view of the subject must become fundamental. If I were asked the question " What is mathematics? " I could find no better answer than these words of Gibbs, though perhaps, lacking his superb control of his tongue, I might be tempted to qualify and

amplify this definition. The constant, almost daily work of trying to see how best to convey mathematical ideas from one mind, through the medium of print, to another, leads to a definition of mathematics, not indeed entirely comprehensive but near to the heart of the matter. Mathematics is the language of abstract rational thought. If reproach is levelled against such high-brow adjectives, it may be remarked that even the most fanatical devotee of "practical" mathematics can hardly deny that in determining, say, the area of a rectangle, he has to form the product of two numbers, an abstract process employing abstract ideas, and that, if he should be anxious to determine the true area, he will do well to adhere to rational methods of multiplication. Mathematics is the natural mode of expression for abstract logical thinking.

But, it may be said, is not this assigning to mathematics a role too simple, too elementary, to be consonant with the dignity of the subject; is it not merely turning the question of what mathematics is into the more trivial question of what mathematics does? No doubt a refined philosophy would be needed accurately to distinguish between the two states of being and doing, but to the first suggestion, that the role proposed is too simple, too trivial, I can give a confident and firm denial. How many times has a colleague from the physics or engineering department come to me with a problem: "Just a bit of elementary mathematics," he says, "you will be able to tell me straight off what I need." Touching faith! We get down to work, and the first task clearly is to turn the question from the physical terms in which he has naturally and properly couched it into the mathematical terms in which I can comprehend it. Eventually we arrive at an abstract formulation of the problem, to which the logical processes of mathematical technique can be applied; but very often at this point the inquirer interrupts: "Oh, you need not do any more now; I have to solve that differential equation, or to evaluate that integral, and I can do that for myself." A question once clearly and precisely formulated in mathematical terms is more than half-way to its answer.

This fundamental linguistic aspect of mathematics presupposes a precise command of our ordinary English language. In ordinary speech it is not always easy to avoid imprecision and vagueness; one of the great merits of mathematics is that it provides a simple method of detecting and exposing such obscurities. But, equally, a statement which is, say, ambiguous, cannot be translated into an unambiguous mathematical statement, it can only be translated into one or other of two precise mathematical statements. Hence the mathematical precision for which we strive can best be attained by precision and clarity in ordinary speech. We must not optimistically leave this matter in the hands of the teacher of English; it is possible that no vagueness, no cheap jargon, is ever tolerated in the English class-room, but what a pupil learns in one class-room he seldom carries to another. Our responsibility as teachers of mathematics is evident; we must not accept standards, either in our pupils or in ourselves, which fall below the good, clear English demanded in the ideal English class-room. For many years I have made a habit of criticising severely the English used by my pupils; I have dealt with any lack of clarity or precision by abuse, ridicule, contempt, whatever mode of severe rebuke seemed most fitted to the offence and to the personality of the offender. Of course I suffer. My pupils soon learn to apply to me the canons by which I judge them; the slightest ambiguity in an examination question is detected and denounced, a lapse into jargon at the blackboard is pilloried without mercy. If in a moment of stress I endeavour to excuse myself by saying, "Well, you know what I mean", the excuse is not accepted; they reply, "Of course we do, but that is no reason why you should not say what you mean." No doubt this discipline is good for me, but that does not matter

much ; what does matter is that it is extremely good for my pupils, who thereby acquire a better insight into the real nature and scope of mathematics than they do in an equal amount of time spent on the polishing of some routine piece of technical manipulation.

Lucid and precise thought is in fact of very little value unless it can be given lucid and precise expression. This is true, not only at the level of the school text-book, but also when applied to the creative work of the front-rank research mathematician. It is no use a mathematician saying that he has proved the famous speculation of Riemann's concerning the zeros of the zeta-function unless he can present his proof to his fellow-experts in a clear and intelligible form. The considered opinion of a great creative mathematician, the late A. E. H. Love, in his Presidential address to the London Mathematical Society *, is emphatic. Speaking of mathematics as a creative art, Love says : " An essential element in the equipment of an investigator is a literary education, or if you prefer it, a training in the means of expression. It is necessary to be articulate, but more than this is desirable. It is desirable to be mathematically articulate . . . to say exactly what one means, neither less nor more. . . . It may be necessary to rewrite a memoir more than once or twice if these advantages are to be secured. It is worth while." That this is true, not only at the high level of which Love was thinking, but at all stages of mathematical thought and teaching, may be a platitude, known and accepted by all, but my experience leads me to believe that it is often forgotten or ignored. How many contributions offered to the *Gazette* contain good and valuable ideas, marred or hidden by crude, careless and inaccurate English. The author may be unaware of his shortcomings ; but, if conscious of them, supposes that it is the job of the editor or the compositor to mend his tattered and disreputable phrases. When we admire, as we all must, the limpid clarity and elegant precision of the writings of the great French mathematicians, do we always realise that this perfection does not come unsought or without effort? It is the result of careful training and self-discipline, the fruit of a devotion to and a pride in high standards of French speaking and writing, whereas in this country such devotion and such pride are found far too seldom.

Now please do not jump to the conclusion that I am asking you to write your mathematics, whether it is on the blackboard for the class, or for the *Gazette*, or for a text-book, in the prose of Milton or Macaulay or Bernard Shaw. The ability to write beautiful English is a gift which has been given to very few ; but we can all write good and clear English. And we ought all to try very hard to do so, because we shall thereby be making mathematics much more comprehensible to our pupils, and if we are not prepared to take some trouble for that purpose then we ought not to be teaching mathematics at all. There are no golden rules for our guidance, but one valuable principle is so easy to state that its very simplicity no doubt accounts for the fact that it is so often ignored : each sentence, whether it contains symbols or not, should obey the ordinary rules of English grammar, and should read as fluently and lucidly as any other decent piece of English. Application of this rule becomes difficult at advanced levels of mathematics, but at the school level it is—or should be—a primary consideration. Its efficacy can easily be tested. Take two school texts, dealing with the same subject for pupils at the same level. Read corresponding chapters, and you will probably find that one book is more easily read and more readily comprehended than the other. Now analyse the structure of the two books, and you will discover, nine times out of ten, that the former, the preferable, book has observed the principle just laid down and that the other has not.

* *Proceedings London Math. Soc.*, (2), 15, (1915), pp. 176–188.

The responsibility which rests on us as mathematicians is all the greater because of our advantages. Some years ago, in a review of Bertrand Russell's *History of Western Philosophy*, Harold Nicolson said that it was written in the clear English which nowadays only mathematicians seemed able to provide. There was a note of surprise in Nicolson's comment, quite unnecessarily, since clear thinking and clear expression are the primary virtues of mathematics, and so clear English should come more easily from our pens than from those of our historians or physicists. This privilege should breed, not carelessness, but increased responsibility. The first lesson derived from a long association with printer's ink has been that mathematics is a language, the language of abstract logical thought, and that the teacher, whose duty it is to train his pupils in this language, must first be master of his own mother tongue.

The second lesson I have learned may be put, with slight and possibly pardonable exaggeration, thus: the teacher who is a thorough master of the mathematics he presents to his classes, and of nothing more, is an unworthy member of our honourable profession. Unworthy, because he is, by reason of a self-imposed restriction, less competent, less efficient than he could be.

We are gathered here as mathematicians, and therefore it may be assumed that we believe that mathematics is a subject of primary importance in the school and university, and that our duty is to teach mathematics as well as we possibly can. Our aim should be, not to flood our pupils' minds with gobbets of *ad hoc* information, to be retained there only until they can be disgorged through the common sewer of the examination system, but to develop the mathematical habit of mind, the clarification of thought, more important still the intellectual curiosity which delights in the formulation of precise problems, and most important of all the intellectual honesty which will not be satisfied with a dusty answer. The teacher whose knowledge covers only his prescribed syllabus will inevitably shrink from encouraging a curiosity whose demands he may not be able to meet, and will try to strangle a nascent intellectual honesty which might prompt some pupil to reject statements such as: "There is no number whose square is -1, so we shall call it i" or, "Some pairs of circles meet in two points, some do not, and therefore by the principle of continuity, any two circles have two points in common." Most of us will agree that these strictures apply, if not to us, then at any rate to some of our colleagues who are not here.

Shortly before the war, Professor Hogben, in the course of a most stimulating presidential address to the London Branch of this Association, gave his recipe for good teaching in biology. It was this: forget all about the syllabus and the examination, and simply concentrate on teaching biology until the pupils are caught up in the subject and can take the examination in their stride. Does this golden rule apply to mathematics? Professor Hogben is clearly of the opinion that it does not, for elsewhere in this same address he gives a recipe for good mathematical teaching, which is "to put into the teaching of mathematics something which does not belong to the subject". The difference between the two recipes would seem to be justified, in his opinion, by some emotive defect in mathematical discourse, a reproach which apparently cannot be levelled against elementary biology. Rather than estimate the precise form of emotional appeal which biology possesses for the adolescent, I would prefer to accept Hogben's rule for any school subject: teach the subject first, and examination technique second, if at all. It is a counsel of perfection if you like, but counsels of perfection have their value in this imperfect world, and if we teachers are not prepared to listen to them, I see few other professions likely to do so in this age.

But at this point a critic might be tempted to urge that this after all is merely the prejudiced view of the mathematician, safely enclosed in his

ivory tower, turning his back on the real world, averting his eyes from the world of concrete everyday things. The critic will argue that the appeal, if any, of arithmetic to the child does not or should not lie in the fascinations of number in the abstract, in the curious and stimulating sense of power which command of number can bring, but in other and rarer delights, in the response arithmetic can give to the pathetic craving of the child to master the exotic delights of rates and taxes, to penetrate the seductive labyrinth, more mysterious than that of Minos, where lurk the Commissioners of Inland Revenue. This is just Hogben's argument over again ; the pupil must have his nose always pressed firmly against the grindstone of the so-called real world, his interests must always be related firmly to experience, though whether the experience of the child or of his teacher seems to be a question of little moment.

Is the critic's position a sound one? Let me quote—I do so by permission of the writer—from a private letter I received a year or two ago. " I get a great deal of pleasure in digging out beautiful solutions to miscellaneous problems. As I get older I realise that it is the base uses to which mathematics is put which ruin it as a subject for most students." This is not the mathematician lost in a world of his own abstractions, but the Director of Naval Construction, whose world of battleships and carriers, submarines and landing-craft, is a very real world, and whose interest in the teaching of mathematics to junior members of the Royal Corps of Naval Constructors is a very real and lively interest. Let the critic mark the " base uses which ruin mathematics as a subject for most students ", and cease to accuse the prejudiced and cloistered views of the mathematician.

Let us then agree that our first duty is to teach mathematics as mathematics and for its own sake. Can we succeed if our knowledge is limited to what we propose to expound in the class-room? By no means. We shall find ourselves presenting mathematics as something closed, something beautiful perhaps but with the beauty of a cold inanimate statue, whereas it should be presented for what it is, a living, growing, developing organism. We shall find ourselves discouraging our pupils from thinking for themselves, lest by so doing they are led into asking us questions to which we can make no intelligent reply. We shall find ourselves in danger of presenting arguments or expounding proofs whose flaws may be more evident to the clearer-sighted of our pupils than they are to us. We shall be at the mercy of the educational quack ; we may even accept his mathematical monstrosities as gospel and commit ourselves to the fallacies of the direction theory of parallels, or suppose that because euclidean geometry is true, non-euclidean geometry must be false, or that an algebra in which multiplication is not commutative is necessarily bogus. These are very real dangers which beset the teacher whose horizon is defined by the walls of his class-room.

Among members of this Association, whose proud claim it is that we hold ourselves debtors to our profession, it is not necessary to elaborate this matter in greater detail. We are agreed on the dangers. But a danger which confronts us every day is apt soon to lose its terrors and to be accepted as part of our working conditions, a normal risk in our profession. It is this complacency which we must guard against, a complacency which inclines all of us to believe that the best text-books are still those which we used at school, that the mathematics we learned at the university is the real goods, and that later developments are so much stuff and nonsense. We cannot afford to be thus bounded in a nut-shell, when we might be kings of infinite space.

But, it will be said, mathematics is so wide a field that the ordinary teacher cannot possibly hope to be acquainted with all the developments, and so what I am suggesting is theoretically inviting but practically out of the question.

This fallacy, for fallacy it is, has been delightfully exploded by Mr. Hope-Jones, in another connection. Some years ago, when the desirability of checking numerical work was being strongly advocated, it was urged on the other side that since we cannot find good checks for all formulae, we need not use them for any. In reply, Mr. Hope-Jones remarked that this argument was on a level with that of the man who says "If we cannot clear all slums, why clear any?" or "If we cannot marry all women, why marry any?" The teacher who, rightly observing that he cannot acquaint himself with all the advances of mathematics, draws the conclusion that he is thereby exempt from the need to acquaint himself with any of them is using the same unsound argument. No one suggests that it is possible in these days to take all mathematics for one's parish. The master of the mechanics of a deformable medium may very likely be at a loss in the delicacies of Banach space, the geometer's handling of the finer points of Fourier analysis may be unsure, and so we humbler people need not ape an omniscience to which even our betters make no claim. It is enough if there is one topic, as narrow in frontage as may be, with which we have a genuine acquaintance. More than this we need not try to possess, but anything less than this and we are neglecting our professional responsibilities.

No doubt this demands a certain amount of time, and I have been told that time for it cannot be found in the busy life we all lead today. Too much, it is said, is being asked when it is maintained that we should devote a couple of hours a week to the study of some genuinely mathematical topic. With this plea I have no sympathy; it is likely to be advanced only by the lazy and the leisured—the busy person can always find time for more work. Time then must be found. What topic should one choose? Well, something which caught your fancy in your student days can be pursued afresh; even if you are so unfortunate as to have no recollection from those days of some topic about which you would like to learn more, do not say "There is nothing in mathematics for me". Turn the pages of Courant & Robbins, *What is Mathematics*, or Bell's *Mathematics, Queen and Servant of Science*, or of the elementary mathematical periodicals to be found in a good library (and therefore always to be found in the Association's library) the *American Mathematical Monthly*, *Scripta Mathematica*, the *Mathematics Magazine*, until some odd fact, some curious connection arouses your interest: what has Morley's triangle to do with the three-cusped hypocycloid; how may it be calculated that there are 1152 possible algebras; or, on the historical side, did Cardan really steal the solution of the cubic from Tartaglia, and why should one wish to steal the solution of a cubic; what led Hamilton to say that algebra is the science of pure time; why does Russell assert that George Boole invented pure mathematics in 1854. Any one of a thousand queries such as these will do for a starting point; and to aid further reading, may I with no modesty at all recommend the guidance to be found in the review pages of the *Gazette*. *Gazette* reviewers are particularly careful to avoid making a review an explanation of how much better the author's book would have been if the reviewer had written it, and take great pains to say what the contents are and how they will appeal to the non-expert seekers after knowledge. A steady perusal of the review pages will bring a very wide field of knowledge before the reader's eyes and show him a multitude of possible pathways along which he may run, walk or dawdle as his fancy takes him. To my mind, there is no better way of keeping in touch with the work of the hour.

There is another matter, apparently not immediately connected with my text, so prominently in our minds at this time that I cannot withold a brief reference. There is an increasing gap between the supply and demand for science teachers; competent and cautious judges have assured us that if the

present trend continues then in 20 years there will be no science teaching in the vast majority of our schools. The situation is not quite so serious in mathematics, but it is serious enough to require consideration. I have no simple remedy to propose; probably no simple remedy exists. But I am sure that increases of salary, adjustments of retiring age, improvements in the material conditions, valuable as they may be, are of themselves merely palliatives, and as such unlikely to effect any substantial redressing of an unbalanced situation. In the past, we have drawn the best of our teachers from those who have seen teaching, not merely as one way of scraping a living, but as a vocation. If the vocational aspect of teaching is neglected, then the number and quality of teachers will diminish. You will not expect me to supply here an elaborate structure of specific constructive proposals; but an attempt to supply such proposals will fail, it seems to me, unless based on a two-fold recognition of a principle which is not primarily concerned with material and concrete conditions of service. First, the community must cease to envisage the teacher of mathematics as an illiterate barbarian, or as a twentieth-century witch-doctor dealing in spells and black magic, and must learn to accord him the status proper to the integral and vital part which mathematics is playing in twentieth-century culture, thought and technical development, to accept the mathematician as one whose contribution to the life of the community is important, intelligible and desirable, perhaps even essential. Secondly, the teacher of mathematics must spare no effort to make himself worthy of this status. And this is where we link up with my previous point, reinforcing my insistence that the teacher must not allow himself to lose touch with the living branches of his subject, that he must find time or make time to keep his intellectual limbs supple, to keep his elementary teaching fresh and vigorous by knowing that he is not dealing with the dry bones of a corpse but with the muscles and sinews of a living growing organism. If the community is to recognise the high status of the mathematician, then the mathematician must begin by showing himself fully worthy of that recognition; more than ever before, he must hold himself a debtor to his profession. Such a two-fold recognition will not cure the present plight; but it will provide a foundation for a cure, and no cure will prosper unless it is based on this foundation.

Finally, one very crude consideration. Had I been told, when a young member of the Association, that printing costs money, I should no doubt have given a light assent to so obvious and trivial an assertion. I should not now regard the assertion as trivial and my assent would be given not lightly but with grave and even passionate conviction. It is no longer sufficient to ask, of a new manuscript, "How good is this?"; the other question must also be asked, "How much is this going to cost?" If, on the other hand, members are inclined to ask what they get for their money, some arithmetical details may be appropriate. In 1903, the annual subscription was 10s., the annual size of the *Gazette* about 150 pages, the amount of material issued to members in the form of reports quite small. Today the annual subscription is 21s., nominally double what it was in 1903 but in actual value very much smaller than the 1903 subscription, the *Gazette* is twice as big, and the annual output of report material is about equal to the size of the 1903 *Gazette*. Of course this has been possible mainly by the great increase in our membership figures, but this is not the whole story. In our annual statement of accounts, it will be seen that the cost of printing the *Gazette* and the Reports per annum is about half as much again as our income from subscriptions. This again is not the whole story, for if it were we should soon have no Association at all. Fortunately we are able to sell to non-members about enough of our various publications to make the Reports self-supporting in the long run, and to make a fair contribution to the cost of the *Gazette*. But it is clear that our steady motion

is not necessarily stable. Had it not been for the great assistance the Treasurer has derived from the covenant scheme, we should today be in a very unhappy position.

What practical lessons are to be learned here? First, that we should all try to enrol new members, not merely on the grounds that the Association has much to offer but also because membership is an obligation which we owe to our profession. Secondly, if we pay an annual subscription, our 21s. will be worth something like 35s. into the Association's income if we take the small additional trouble to make a covenant with the Association, details of which will be gladly supplied by the Treasurer. Life members, who cannot help in this way, and who may be congratulating themselves on their forethought in making a life composition from which they now score heavily, may relieve any twinges of conscience by helping to swell the figures in the column in which the Treasurer records donations.

One last word. To the Treasurer and Secretaries of this Association I owe most grateful thanks for the devoted, unostentatious service by which, at no small self-sacrifice, they cause our affairs to flourish ; on my own behalf, and on behalf of all our members, I tender them our sincere gratitude.

T. A. A. B.

GLEANINGS FAR AND NEAR

1770. FERMAT OUT-MODED.

Mathematical Problem No. 7. Find a pair of numbers such that their sum is a perfect square, the sum of their squares is a perfect square, and the sum of their cubes is a perfect cube.—*Journal of the Incorporated Association of Assistant Masters in Secondary Schools*, May 1953. [Per Mr. G. Tyson.]

1771. An application to birth statistics when one assumes that parents have boys with probability $\frac{1}{2}$ but tend to stop after the first boy is discussed.—*Math. Reviews*, XIV (5), p. 485 (1953). [Per Mr. W. A. O'N. Waugh.]

1772. They stared at that other thermometer which showed how winter lived in this snowing ship. One thousand degrees below zero.—Ray Bradbury, " The Golden Apples of the Sun ", *The Observer*, 27th September 1953. [Per Mr. J. E. Blamey.]

1773. Medicine and surgery : Batchelor, G. K., *The theory of homogeneous turbulence.—Books of the Month*, June and July, 1953. [Per Mr. S. H. Clarke.]

1774. That cut from Hutton bisected the slips.—B.B.C. commentator, 5th Test Match, 1953. [Per Mr. H. V. Lowry.]

1775. The spectacle made me giddy, and I had to move or fall. The wall was not quite perpendicular, but as far as I could see a slope of about sixty degrees.—John Buchan, *Prester John*, Ch. XXI. [Per Mr. S. B. Rooum.]

1776. BOWDLERISATION?

Entry in the Table of Contents, *Combinatory Analysis*, Vol. II, P. A. Macmahon (Cambridge Press, 1916) :

" Art. 250. Sylvester's more refined correspondence . . . p. 14."

No reference is given to the rest of Sylvester's correspondence. [Per Mr. R. A. Fairthorne.]

SOME PROPERTIES OF PURE RECURRING DECIMALS.

By J. S. Batty.

1. The earlier sections of this account are concerned with some standard properties of pure recurring decimals. These include the cyclic property, illustrated for example by the decimals representing the six proper fractions with denominator 7, each of which has a period using a cyclic permutation of the digits 142857. It was a question concerning this property, raised by a student, which led to the writer's interest in the properties considered in the later sections. References are given in Dickson's *History of the Theory of Numbers*, Vol. I, Ch. 6.

In such works as *An Introduction to the Theory of Numbers*, by Hardy and Wright, the proofs given of the standard properties of decimals naturally assume some knowledge of number theory. The present account is self-contained, but it is convenient to use the congruence notation, *i.e.* $a \equiv b$, mod q, implies that $a = b + nq$, where n is an integer. No properties of congruences are used except such as follow at once from this definition.

2. A pure recurring decimal* is generated by a reduced proper fraction p/q in which q is prime to 10. If q is not prime to 10 the decimal begins with a non-recurring portion and is not "pure"; thus $1/6 = \cdot 1\dot{6}$. If q has only the factors 2 and 5, the non-recurring portion is followed by 0's only, or 9's only, and may alternatively be said to terminate†. We consider pure recurring decimals only; thus *p is prime to q and less than q, and q is prime to* 10.

If $p_1/q = \cdot \dot{a}_1 a_2 \ldots \dot{a}_\lambda$, the process of generating this decimal by successive divisions may be summarised by the equations

$$10p_r = a_r q + p_{r+1}, \quad r = 1, 2, \ldots, \lambda, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $p_{\lambda+1} = p_1$ and $0 < p_r < q$.

When we are not concerned with the digits a_r these equations may be replaced by the congruences to the modulus q:

$$p_{r+1} \equiv 10p_r \equiv 10^r p_1, \quad r = 1, 2, \ldots, \lambda. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The set $\{p_r\}$ will be called a *linked set* of remainders of q. If $q = 7$, $\lambda = 6$, and $\{p_r\} = \{1, 3, 2, 6, 4, 5\}$; $q = 13$ gives $\lambda = 6$, and there are two linked sets

$$\{1, 10, 9, 12, 3, 4\} \quad \text{and} \quad \{2, 7, 5, 11, 6, 8\}$$

A cyclic permutation of the equations (1) may be used to bring any p_r into the leading position; thus

$$p_r/q = \cdot \dot{a}_r \ldots a_\lambda a_1 \ldots \dot{a}_{r-1}.$$

For example:
$$\tfrac{1}{7} = \cdot \dot{1}42857\dot{1}, \quad \tfrac{3}{7} = \cdot \dot{4}2857\dot{1}\dot{4};$$

$$\tfrac{1}{13} = \cdot \dot{0}76923\dot{0}, \quad \tfrac{12}{13} = \cdot \dot{9}23076\dot{9}.$$

The λ repetends formed by these cyclic permutations will be regarded as the same "period"; λ is the *length* of the period.

A fundamental property of the length of the period is

(2·1) λ is the smallest value of x for which $10^x \equiv 1$, mod q.

For (2) with $r = \lambda$ gives

$$p_{\lambda+1} = p_1 \equiv 10^\lambda p_1, \quad \text{mod } q,$$

* Decimals to the base 10 are used throughout; the changes necessary for a general base, prime to q, are trivial.

† Hardy and Wright, Ch. 9.

and p_1, being prime to q, may be cancelled. The result follows since $p_r \neq p_1$ for $r \leqslant \lambda$.

An immediate consequence is that λ *depends only on* q. We may therefore say that "q has period of length λ".

3. The number of integers less than q and prime to q, or the *residues* of q, is denoted by $\phi(q)$, Euler's function. Since the complete set of residues of q contains any linked set, $\lambda \leqslant \phi(q)$. For prime q, $\phi(q) = q - 1$; as examples of the corresponding values of λ we have $q = 11$, $\lambda = 2$; $q = 13$, $\lambda = 6$; $q = 17$, $\lambda = 16$; $q = 31$, $\lambda = 15$.

We now show that, as illustrated by these examples,

$$\lambda \text{ divides } \phi(q). \tag{3·1}$$

Choose any residue of q as p_1, and form the linked set $\{p_r\}$ as in (1). If this set contains all residues of q, $\lambda = \phi(q)$. If not, choose a residue p_1' not in $\{p_r\}$ and repeat; and so on until all residues are used. Thus the $\phi(q)$ residues are separated into sets of λ, *i.e.* λ divides $\phi(q)$.

The quotient $\phi(q)/\lambda$ is the number of *distinct* periods of q. If $\lambda = \phi(q)$, q has one period only; it can be shown that this is possible only if q is a prime or a power of a prime. Primes with one period only have been called "proper" primes*; thus for a proper prime $\lambda = q - 1$. The first ten proper primes are 7, 17, 19, 23, 29, 47, 59, 61, 97, 109. There are 60 among the 127 primes between 3 and 1000.

The proofs usually given† of (3.1) depend on the generalised Fermat Theorem. Alternatively Fermat's Theorem may be deduced‡ from (3.1); for, by (2.1)

$$10^{n\lambda} \equiv 1, \mod q,$$

for any integer n; (3.1) then gives

$$10^{\phi(q)} \equiv 1, \mod q.$$

Replacing the base 10 throughout by any integer a prime to q, we have

$$a^{\phi(q)} \equiv 1, \mod q,$$

which is the generalised Fermat Theorem.

If q is prime, the result becomes

$$a^{q-1} \equiv 1, \mod q,$$

for a prime to q, which is Fermat's Theorem.

4. In the sequel we examine properties of sums of remainders and of digits.

(4.1) If q is not a multiple of 3, the sum of all remainders in a linked set is a multiple of q, and the sum of the corresponding digits is the same multiple of 9. Thus

$$\sum_{s=1}^{\lambda} p_s = qK, \quad \sum_{s=1}^{\lambda} a_s = 9K$$

where K is an integer.

The examples $q = 7$ and $q = 13$, cited earlier, each give $K = 3$.

To prove (4.1), we add the equations (1), giving $9\ \Sigma p = q\ \Sigma a$. If q is not a multiple of 3, it follows that $\Sigma a = 9K$ for some integer K, whence $\Sigma p = qK$.

It is easily seen that for all proper primes, $K = \frac{1}{2}\lambda$. For, if q is proper,

* Dickson, p. 166.

† For example, Hardy and Wright, Ch. 9; Ore, *Number Theory and its History*, Ch. 13.

‡ Dickson, p. 167.

$\lambda = q-1$, and $\Sigma p = 1+2+3\ldots+(q-1) = \frac{1}{2}q(q-1) = \frac{1}{2}\lambda q$. More general conditions giving $K = \frac{1}{2}\lambda$ will be found in § 7.

We now prove the following result, concerning subsets of the sets $\{p_r\}$ and $\{a_r\}$:

(4.2) If m, n are any complementary divisors of λ, and if q, $10^m - 1$ are coprime, then for $r = 1, 2, \ldots, m$,

$$\sum_{s=0}^{n-1} p_{r+sm} = k_r q,$$

$$\sum_{s=0}^{n-1} a_{r+sm} = 10k_r - k_{r+1},$$

where k_r is an integer satisfying $1 \leqslant k_r \leqslant n-1$, and $k_{m+1} = k_1$.

If $m = 1$, this reduces to (4.1), K being the common value of k_1 and k_2. If λ is prime, this is the only case.

To prove (4.2), we write

$$10^\lambda - 1 = 10^{mn} - 1 = (10^m - 1)(10^{(n-1)m} + \ldots + 1)$$

$$= (10^m - 1)\sum_{s=0}^{n-1} 10^{sm}.$$

Now, by (2.1), q divides the left-hand member and does not divide $10^m - 1$. Thus, if q and $10^m - 1$ are coprime, it follows that q divides $\sum_{s=0}^{n-1} 10^{sm}$ and therefore also $p_r \sum_{s=0}^{n-1} 10^{sm}$. Now (2) gives

$$p_r \sum_{s=0}^{n-1} 10^{sm} \equiv \sum_{s=0}^{n-1} p_{r+sm}, \mod q,$$

for $1 \leqslant r \leqslant m$. Hence q divides the right-hand member, *i.e.*

$$\sum_{s=0}^{n-1} p_{r+sm} = k_r q, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

for some integer k_r. Also, since $0 < p_r < q$,

$$1 \leqslant k_r \leqslant n-1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

From (1) and (4.1) we have, after cancelling q,

$$\sum_{s=0}^{n-1} a_{r+sm} = 10k_r - k_{r+1}, \quad 1 \leqslant r \leqslant m, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where $k_{m+1} = k_1$.

An example is $q = 31$, $\lambda = 15$. One linked set of remainders is

$$\{1, 10, 7, 8, 18, 25, 2, 20, 14, 16, 5, 19, 4, 9, 28,\}$$

and

$$\tfrac{1}{31} = \cdot\dot{0}32258064516129\dot{0}.$$

Taking $m = 5$, $n = 3$, gives for $r = 1$ and $r = 2$,

$$\sum_{s=0}^{2} p_{5s+1} = 1 + 25 + 5 = 31,$$

$$\sum_{s=0}^{2} p_{5s+2} = 10 + 2 + 19 = 31,$$

$$\sum_{s=0}^{2} a_{5s+1} = 0 + 8 + 1 = 9,$$

$$\sum_{s=0}^{2} a_{5s+2} = 3 + 0 + 6 = 9.$$

Interchanging m and n gives

$$\sum_{s=0}^{4} p_{3s+1} = 1+8+2+16+4 = 31, \quad (k_1 = 1),$$

$$\sum_{s=0}^{4} p_{3s+2} = 10+18+20+5+9 = 62 = 2 \cdot 31, \quad (k_2 = 2),$$

$$\sum_{0}^{4} a_{3s+1} = 0+2+0+5+1 = 8 = 10k_1 - k_2.$$

5. The relations of (4.2) will now be used to prove

(5.1) If M is the greatest (proper) divisor of λ, such that $10^M - 1, q$ are co-prime, then for any linked set of remainders of q, the value of K as defined by (4.1) satisfies the inequality

$$M \leqslant K \leqslant \lambda - M.$$

For, if λ is composite, any divisor m (satisfying the co-prime condition of (4.2)) may be used to separate the set $\{p_r\}$ into m subsets, each of λ/m terms; and for the corresponding set $\{k_r\}_m$ (4.2) gives $1 \leqslant k_r \leqslant \lambda/m - 1$. Summing over the m sets gives

$$m \leqslant K \leqslant \lambda - m.$$

Since K satisfies this inequality for each m,

$$M \leqslant K \leqslant \lambda - M, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where M is the greatest (proper) divisor of λ.

If K has its least value M, then in $\{k_r\}_M$ each k_r has its least value, unity; thus with $\lambda = NM$ we have

$$\sum_{s=0}^{N-1} p_{r+sM} = q, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$\sum_{s=0}^{N-1} a_{r+sM} = 9. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

As an example, $q = 333667$ gives*

$$1/q = \dot{0}00002997\dot{1}, \quad \lambda = 9, \quad \Sigma a = 27, \quad K = M = 3, \quad \text{and}$$
$$a_1 + a_4 + a_7 = a_2 + a_5 + a_8 = a_3 + a_6 + a_9 = 9.$$

Similarly, if K has its greatest value $\lambda - M$, each k_r in the set $\{k_r\}_M$ is $\lambda/M - 1$; an example is given by $(q-1)/q$, with q as above.

6. If λ is even, $M = \frac{1}{2}\lambda$; (6), (7) and (8) give

$$K = \tfrac{1}{2}\lambda \ , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$p_r + p_{\frac{1}{2}\lambda + r} = q, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$a_r + a_{\frac{1}{2}\lambda + r} = 9. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Now, two remainders with sum q and two digits with sum 9 may be called *complementary*; the above equations state that *corresponding remainders and digits in the two half-periods are complementary.* Periods exhibiting these complementary properties will be said to belong to the class " C ". A period of class " C " is necessarily of even length; a further condition, which is sufficient, is that q and $10^{\frac{1}{2}\lambda} - 1$ are coprime. These conditions depend only on q; therefore *if one period of q is of class " C " so are all periods.* In such a

* This is the only with period 9, the smallest odd composite period.

case we may say that q is of class " C ". It follows from (2·1) that all primes or powers of primes with even period are of class " C ". In particular all " proper " primes are of this class and also those prime powers q^γ for which $\lambda = \phi(q^\gamma)$, since $\phi(q^\gamma)$ may be shown to be $q^{\gamma-1}(q-1)$, which is even for prime $q > 2$.

The complementary property of the digits in the half periods gives at once :

(6.1) The sum of the two half-periods of a decimal of class " C " is $99\ldots9$ to $\frac{1}{2}\lambda$ digits, or $10^{\frac{1}{2}\lambda} - 1$.

This result will be generalized in § 8.

The following examples of periods of class " C " are given by the first three " improper " primes of the class, and one prime power.

$q = 11$: 5 periods : 09, 18, 27, 36, 45 ;

$q = 13$: 2 periods : 076923, 153846 ;

$q = 73$: 9 periods :

01369863, 02739726, 04109589,
34246575, 05479452, 12328767,
06849315, 16438356, 08219178;

$q = 49$: 1 period : 020408163265306122448979591836734693877551.

An example of the possible failure of the complementary property, and others, when q and $10^{\frac{1}{2}\lambda} - 1$ have a common factor, is given by $q = 1111$, which has the factor 11 with $10^2 - 1$, and one of the 25 periods is 0001.

7. The characteristic features of a period of class " C " are that λ is even and the complementary remainder $q - p_r$ occurs $\frac{1}{2}\lambda$ stages after p_r in the division process.

Conversely, it is clear that if the remainder at, say, the mth stage is $q - p_1$, then the corresponding digit is $9 - a_1$, and the process continues with each of the first m remainders and each of the first m digits replaced by its complement, until the remainder p recurs after a further m stages. Thus $\lambda = 2m$, and the period has two complementary halves.

If the remainder $q - p_1$ does not occur at any stage then p_1 and $q - p_1$ belong to distinct linked sets, the periods for p_1/q and $(q - p_1)/q$ are distinct, and corresponding digits in each are complementary, the sum being $\cdot\dot{9} = 1$. The various periods separate into complementary pairs ; the number of these periods, $\phi(q)/\lambda$, is therefore even.

Summarizing these results :

A necessary condition for q to be of class " C " is that λ is even. A sufficient condition is that $10^{\frac{1}{2}\lambda} - 1$ is an integer prime to q. Another sufficient condition is that $\phi(q)/\lambda$ is odd.

If q is not of class " C " its periods form complementary pairs. If λ is even, a necessary condition is that $10^{\frac{1}{2}\lambda} - 1$ has a common factor with q ; another necessary condition in this case is that $\phi(q)$ is a multiple of 4.

Examples are :

$q = 41$: 8 periods forming pairs :

02439, 95121, 07317, 14634, $(K = 2)$,
97560, 04878, 92682, 85365, $(K = 3)$;

$q = 21$: 2 periods : 047619, $(K = 3)$,
952380, $(K = 3)$.

For $q = 21$, the coprime condition is not satisfied, 21 and $10^3 - 1$ having the common factor 3. Although the complementary property fails, other properties still hold ; thus $\Sigma a = 9K$, and $K = \frac{1}{2}\lambda$.

When periods occur in complementary pairs the sum of all digits in the pair is 9λ; thus the two values of K for the pair are complementary with sum λ, as illustrated in the above examples.

8. The following result is a generalisation of (6.1).

(8.1) If λ is composite with a divisor m, then

$$S_m = a_1 a_2 \ldots a_m + a_{m+1} \ldots a_{2m} + \ldots + a_{\lambda-m+1} \ldots a_\lambda$$

is a multiple* of $10^m - 1$.

For $S_m = 10^{m-1}(a_1 + a_{m+1} + \ldots a_{\lambda-m+1}) + 10^{m-2}(a_2 + \ldots a_{\lambda-m+2})$
$+ \ldots a_m + a_{2m} + \ldots a_\lambda$
$= 10^{m-1}(10k_1 - k_2) + 10^{m-2}(10k_2 - k_3) + \ldots 10k_m - k_1$, by (5)
$= (10^m - 1)k_1$.

Similarly, if the digits are grouped cyclically starting at a_r the sum is

$$(10^m - 1)k_r.$$

As illustrations we have the first two primes with odd length of period.

$q = 31$. Periods are :

(1) 032258064516129, $(K = 6)$.
(2) 967741935483870, $(K = 3)$.
(1) $S_3 = 032 + 258 + 064 + 516 + 129 = 10^3 - 1$,
$S_5 = 03225 + 80645 + 16129 = 10^5 - 1$.
(2) $S_3 = 2\,(10^3 - 1)$,
$S_5 = 10^5 - 1$.

$q = 43$. Periods are :

(1) 023255813953488372093, $(K = 10)$.
(2) 976744186046511627906, $(K = 11)$.
(1) $S_3 = 3\,(10^3 - 1)$,
$S_7 = 10^7 - 1$.
(2) $S_3 = 4\,(10^3 - 1)$,
$S_7 = 2\,(10^7 - 1)$. J. S. B.

* If (5.3) is satisfied this result is trivial, the multiple being unity.

1777. The first time I ever went to Lochearnhead was with two stalwart mathematicians addicted to that curious deflection of the sporting instinct, mountaineering. It is rather strange that mathematicians, as a class, seem to have an irresistible urge towards activities beginning with the letter M, such as music and mountaineering. To me the perfect image of a mathematician is a man taking a small class in algebra, at night, on the top of Bear Mountain.—Wilfred Taylor, *Scot Free*, p. 32.

The Rev. C. L. Dodgson might have added " mouse-traps, and the moon, and memory, and muchness ". [Per Mr. Sydney Thomson.]

1778. I advise the young man not to despise mathematics (with which we may also reckon music), but he ought not to devote too much time to this subject. It is useful to those who know it and an obstacle to those who do not. But it does not yield any great profit to those who grow old in its service, and they are reduced to wandering from place to place in order not to perish for lack of activity.—Zwingli, *Of the Upbringing and Education of Youth in Good Manners and Christian Discipline*, 1528, translated by G. W. Bromiley (S.C.M., 1953). [Per Mr. J. W. Ashley Smith.]

WHY TEACH MATHEMATICS?

By John Maclean

1. *Methods, Matter, Aims.*

The question about the teaching of Mathematics considered most frequently and with most detail is *How?* This is the main preoccupation of the Mathematical Association Reports on separate subjects: it is of special interest to note that in the Report on the Teaching of Calculus oft repetition of the phrase "the teacher will know" implies that the topic is as rich as human nature. Yet for writers in the *Mathematical Gazette*, and for most scholarship winners, the question is superfluous, though all appreciate a good teacher. (For these too the answer to *Why?* is simple—the urge of joy or duty.) On the other hand, it would scarcely be denied that for more than two-thirds of the pupils in our schools the question *How?* is unanswerable under present conditions: for them we are attempting the impossible.

Much attention has had to be given to *What?*; for the failures and waste from the attitude of a century ago and later to the content of school mathematics became intolerable as society moved towards more general education and freedom. That a training in Euclidean method was at all deemed right was partly due to misapprehension as to what Euclid had really done, but mainly to the fact that able mathematical pupils actually enjoyed much of this magnificent discipline. It is still a specially dangerous temptation to teachers of mathematics to present pupils with what appeals to the teacher personally. Dispersion of ability and of appreciation is greater in mathematics than in any other subject, and it is essential to ask continually how far it is right to be content with success in teaching the minority of "better" pupils, certainly not with securing scholarships. There was also conservatism in the interpretation of "mathematics", another feature which has to be watched even now.

Why? is actually more difficult and has been answered less frequently, notably in the Spens Report (1938) in terms of "the main lines which the creative spirit of man has followed". This has been often quoted, and made more definite (as "to get as far as possible along the road ... to abstract thought", *Gazette*, XXXI, 20). But to one who comes from outside the English system it is not clear that this answer has been effective. Our reaction against tradition is not strong, and we have not set ourselves very definitely to characterise for school use the lines the creative spirit is now following. Thus, for Secondary Modern Schools (*Gazette*, XXXV, 145) interest is the aim, but this, though necessary, is scarcely sufficient.

These questions *How? What? Why?* are of course interdependent. The emphasis in this article will be on aspects of the needs of children of today and tomorrow, an emphasis sufficient, it may be, to show that the question *Why?* needs to be considered afresh. It will also be urged that a time has come to change the setup of the Mathematical Association in its effort to ensure effective teaching.

2. *As We Are.*

First it will be well if we, as our predecessors were forced to do, ask in what degrees we are succeeding in our educational task. A few mathematical teachers, especially in colleges, may be content that the important work of training able pupils is being done well, and that others are not suffering unduly. More generally, the answer would be: "We are doing all we can: it is (necessary and) sufficient to meet the demands of the syllabus, which has been recently revised, and question papers have been greatly improved." This

ignores the Secondary Modern Schools, the problem in which is probably the most urgent today if the educational climate is to improve; but let that pass meantime. On the other hand there is a persistent sense of failure, notably in regard to training in the will o' the wisp, accuracy, and in alertness, which makes not a few teachers almost despair. ("Some of the arithmetic done by G.C. candidates, good ones at that, is appalling", etc.) Let us consider a phenomenon which may take us deeper into the nature of our failure and open out fresh aspects of it. No matter that the example comes from America, for, notwithstanding the decimal system, the same blights affect the study of "Mathematics" there. In an interesting and noteworthy book, *The Economics of Employment* (1951), the counsel is given to those who do not find diagrams helpful to disregard references to where they are presented. Further "It is possible to skip the algebra. . . . The only purpose of this algebra is to make the text easier to follow. Readers who are allergic to algebra are hindered rather than helped and should skip it." Then, on page 69, "We have two equations (1) $Y=C$, (2) $C=40+\frac{2}{3}(Y-40)$", and after half a page it is found that $Y=40$. Then for another half page: "Or, without algebra, we can see without much difficulty that it is impossible for any other income than 40. . . ." It should be said that this logic (it fits, therefore, it is true) is typical of the non-algebraic treatment: it is intensely arithmetical and demands ability as well as extreme patience. What has happened to the teaching of algebra that this procedure is thought best by gifted economists? We have been cautioned in the *Mathematical Gazette* (XXIII, 20) against making easy things difficult, but it would be difficult to find in a mathematics textbook anything so perverse as is set out on, say, p. 94 of this economics book. Briefly, it has been established, rather dogmatically, that the equations $C=40+\frac{2}{3}(Y-40)$, $Y=C+I$ are satisfied by $C=60$, $I=10$, $Y=70$. What would be the effect of making $I=11$? It is first suggested that $Y=71$ will not do, nor even $Y=71\frac{2}{3}$: this would "lead to more C and this would lead to still more Y and so on to still more C and so on". "The answer is that Y will keep on increasing until it reaches the level where a gap equal to the additional I is provided between Y and C" and then a paragraph of arithmetic to show that substitution of $Y=73$, etc. does the trick. Next comes a paragraph to show an alternative psychological viewpoint involving $S=I$. It may be right and illuminating to look at the flowers in the garden from different viewpoints though one cannot describe how they grow. And in particular it *may* be unjust to complain that the "geometrical" series implied in the "and so on"s quoted is not made explicit as

$$C=\tfrac{1}{3}\times 40+\tfrac{2}{3}\times 70+\sum_0^\infty(\tfrac{2}{3})^r=60+\tfrac{2}{3}\times 3=62;$$

for the book contains careful and excellent diagrams which however are given only a meagre description. In view of the appeal the "cobweb" form of iteration makes to economists an opportunity was missed of displaying in these diagrams a simple "staircase" iteration (as on p. 269 there) corresponding to the series hinted at. But all this would be worth nothing compared to a straightforward solution of the very simple equations. It is a sad comment on the teaching of the simplest algebra that books of first-rate clarity on a quantitative subject should have to be written in such a wasteful way for "educated" readers.

Nor are we likely to get comfort from what another lucid writer says, though he believes in being firm with his readers and puts the blame elsewhere than on the makers of mathematics syllabuses. He writes "(Note to diagram-haters; it pays to overcome your loathing when disentangling this side of cost theory)" (Croome: *The Candidates' Companion*, p. 72). Costs are listed as overhead, prime, unit, total, marginal, average, variable; and

so we are not here concerned with highly sublimated economics. But things are complex enough. Note too a sentence from Cambridge. It is from the preface (page v) to recent lectures by Pigou on Keynes : " Where the argument is in essence mathematical it has been translated into words."

Similarly in other directions. Coulson, in his *Valence* (1952), writes of " Those who suffer from an innate fear of mathematics ". Can any teacher of mathematics, even one taking the narrowest view of his task, get any comfort from such remarks? The indications are that eager minds in our classes are being dulled, certainly denied what they are to need.*

3. *Glimpses of the New Scientific World.*

Now let us look, as squarely as we can in brief space, at some aspects of the mathematical demands that are beginning to be made on some important classes of students in colleges. To continue with the position which now faces our young economists, the debate on the need and usefulness of mathematics in the study of economic science was carried to a new stage in the authoritative statement of Samuelson in his *Foundations of Economic Analysis* (1947). On p. 6, " I have come to feel that Marshall's dictum that ' it seems doubtful whether anyone spends his time well in reading lengthy translations of economic doctrines into mathematics that have not been made by himself ' should be exactly reversed. The laborious literary working over of essentially simple mathematical concepts such as is characteristic of much modern economic theory is not only unrewarding from the standpoint of advancing the science but involves as well gymnastics of a peculiarly depraved type. . . . My own interest in mathematics has been secondary."

More detail is given in a quotation from the *Economic Journal* for December, 1950, p. 675 : " The theory is expressed mathematically in terms of derivatives, i.e. rates or directions of change. Most economists, however, cannot operate at all easily in these terms. In expounding and applying the theory they throw it into diagrammatic form, in two (or perhaps in three) dimensions, and allow the hypothetical customer to jump from one situation to another, as represented by two distinct points in the diagram . . . difficulties in two ways. . . . These difficulties arise solely because of the attempt to translate the fundamental equations into diagrammatic terms with finite changes in prices and income." For the strange confusion in the situation we may look at a review in the same issue (p. 788) of a book on Statistics in its 14th edition. It is referred to as " suitable for students with a rare and somewhat improbable combination of knowledge and ignorance. It is very doubtful whether many aspiring statisticians, equipped with the necessary mastery of algebra and analytical geometry (not to mention determinants!) required to cope intelligently with this book, are at the same time entirely unacquainted with the calculus. Nor is it true that because ideas in algebra are in general simpler than those in the calculus, all algebra is necessarily easier than all calculus. A judicious and restrained use of calculus accompanied by some simplification of the algebra would do much to clarify some of the arguments in this book ".

At a more elementary level the complaint about unnecessary obscurity is often made. Thus Bertrand Russell broadcast (*Listener*, 14.6.51, p. 954) " The science of economics has been wrapped around by the theorists in a

* This is stated explicitly for the case of university students of engineering by Sir David Pye in his Presidential Address to the Institution of Mechanical Engineers (*Proc.*, Vol. 166A, 1952, p. 269); and it is implied in a most noteworthy review of a a treatise on Physical Chemistry in (of all places) *Mathematical Reviews* for June, 1953, p. 593.

series of many veils, which have caused the plain man to suppose that there must be something indecent about its naked form ", and so he proceeds " to begin at the beginning ". But the " plain man " makes no use of mathematics in his task, and is not conspicuously successful in revealing the truth about simple transactions. Thereby hangs a tale of modern economics ; and we shall simply remark that the part that conservatism plays in causing waste and confusion may be strangely seen too in a recent " Preparatory Mathematics for Engineers " which actually gives the formerly usual procedure for finding square roots. This is fascinating for some minds—a non-mathematical friend likes to tell me how he perplexed mathematical experts by demanding that cube roots be found similarly, as demonstrated to him in his youth! But this is going back to *As We Are*, certainly not *As We Should Be*.

Turning to a very different region of application we note the remarkable statement by Sir Henry Dale (*Brit. Med. J.*, 25.11.50, p. 1189) in lecturing on Scientific Method in Medical Research : " If I were asked to choose one other among the more fundamental scientific activities as a competitor with biochemistry in these recent decades, for depth and range of influence on thought and practice in medical research, I should name the science of statistics. Quietly, but irresistibly . . . statistical methods and principles have been exercising and establishing their corrective influence . . . in ward and laboratory ". An unexpected instance of this is in chromatography treated in a thorough way—at least, when going to the second of three lectures on chromatographic analysis, I did not expect to hear a discourse on frequency, especially normal, distributions. The audience too looked quite unprepared for this and some actually fled before the end. In fact, all that was said could have been presented more effectively to the sixth form and even earlier in school. Discussion is now taking place in London as to what statistics can be taught for the First Professional in Medicine. I suspect that no real soluton will be obtained before satisfactory mathematical foundations are laid in the schools : then it would be easy to introduce statistical devices as needed. A professor of epidemiology writes on the mathematical needs of medical workers : " The present position that literally 99% of medical men shudder when presented with mathematics of matriculation standard which they all reached at one time, shows that there is something grossly wrong in the general attitude towards the subject. The remedy does not lie in persuading doctors, it lies in persuading schoolboys that mathematics are something vital and comprehensible which can be used to solve ordinary problems. It would be an exaggeration to say that I can talk about epidemiology in sensible terms to 1% of the postgraduate medical students I come across." This he amplified later by an emphatic statement that what was needed in his view was an understanding of " function ".

It would be easy to enlarge on the waste and vexation in the study of quantitative aspects of many subjects through ignorance of the relevant elementary mathematics, but for our purpose here it is sufficient to show as above that this phenomenon is common. Just two small but important further items will be mentioned to indicate another direction in which emphasis that is absent now should be laid. It is not right that a student of chemistry should meet a logarithmic scale first in connection with p_H or p_{OH} : the way in which ion concentration is set out in some textbooks indicates that the student is scarcely expected really to understand the nature of the variation involved. Again chemists share with economists and others a need to appreciate that " elasticity " is represented by the slope of the tangent to a curve on fully logarithmic rulings.

Note should be taken also within industry and commerce of the tensions, lessening we hope, between the practical commonsense men and the planners

—in a wider sense than for administration: linear programming has now become a mathematical topic! The education of those who will not or cannot see the worth of the longer vista should be considered seriously with a view to at least making them in some degree able to appreciate the problems of the designers: they should be given as rich an idea as possible of interrelated change. (The complementary problem of enabling the planners to get the feel of things is clearer, and is not our concern.)

Protests will be made that such matters do not affect the great majority of pupils, but these protests are based on the assumption that things educational are changed really little. Probably I am not alone in feeling humiliated when accompanying a modern youth round an exhibition of up-to-date mechanical or electrical devices. The only consolation is that quick recognition of the characteristic essential parts of devices is based on familiarity and not on full understanding. When we have learned to question the position that the customary in education is fundamental, we may be ready for a change in our educational time scale which will affect deeper aspects of perception than keen youth can usually modify by their own efforts. But our practical women and men demur! However, it would be premature in what follows to attempt to set out such a scale. When the time comes, "the teacher will know".

4. *Current Reforms.*

Let us examine now some points about the changes we are trying to make, apart from the effort which all approve to make teaching more stimulating. These reforms on the whole, it is agreed, have been an advance, though I think we are still held rather tightly by the past. The Association for the Improvement of Geometrical Teaching (1871) might have paused had it realised that the effect of its efforts would be the banishment of Geometry, as well as of Euclid, from the schools. There are doubtless many who are not ready to acknowledge that this is so, but the rejection of Forder's *Geometry for Schools* (1930) was probably the writing on the wall, or at least the sign that Geometry for Schools was an impossiblity. When teachers now look at one another and begin to tot up the number of theorems that recent question papers demand should be taught, it is time to ask whither, intentionally or not, we have been led. Even were I competent to do so, it is not necessary to discuss this matter here; I am content to note the contrasts in time and to affirm that the sooner Geometry ceases to be a full subject the readier shall we be to enter into the freedom that will benefit even the best of our pupils. The practice of the Mathematical Association has not been very consistent in this respect: it misses no opportunity to give a verbal blessing to unifying the presentation of mathematics, but it continues apparently content with preparation of reports on separate subjects—convenience is scarcely a sufficient explanation. It is true that our Interim Reports on mathematics in the new secondary schools are unified, but we were told at the Annual Meeting that the Report for Secondary Modern Schools had not elicited a large response, and so there is some reason here for discontent.

It may be useful to look more closely at the Report on the Teaching of Calculus in Schools (1951), though not because it should come nearest to emphasising the importance of functions. In large degree the Report has been anticipated by some very good textbooks on Calculus, but incidentally it contains three statements which need more emphasis. They occur in Chap. 3 on Introductory Work. The first (*A*) is in 3.0: "it is not desirable to postpone the start of the calculus in order to try to deal with all the dfficulties that may possibly arise later". This sounds disastrously in contrast with the commonsense logical cautions, "One step at a time", "Have all your tools ready" "We must make sure first"—essential to the establishment of confidence

during junior school experiences, but a hindrance if carried too far. The ways in which truth is apprehended are many. After all, this counsel of the Report is but the old saying of Chrystal, or older, " Every mathematical book that is worth anything must be read backwards and forwards . . . the advice of a great French mathematician, ' Allez en avant, et la foi vous viendra ' ". That the point is far-reaching is shown by the fact that, even for advanced work in hydrodynamics, Garrett Birkhoff in 1950 deemed it worth while to write, " Science was not built by timid or faltering hands ".

The second statement (B) is about the unifying of the teaching of mathematics, mooted more than a score of years ago (*Math. Gaz.*, XII. 327), urged in detail in the Spens Report (p. 237), and reaffirmed by the 1941 Norwood Report (p. 106–7), yet a cause, as noted above, which is not making spectacular progress. In the Calculus Report it is (3.1) called a principle, but there is no mention of its complement (B'), the unifying power and role of mathematics which will become of increasing importance for education generally. As is well known, the same mathematical devices are used in widely diverse situations, but it is their use in fairly closely related subjects (e.g. agriculture and physiology) that can be made to help to counteract the too narrow specialisation which is so widely deplored, and yet which under present conditions seems inescapable. When courses in mathematics are really adjusted to the modern situation, it will be easier at a comparatively early stage (if late in schools) to show (by parallel textbooks) the applicability of a device to a group of subjects, possibly as wide as biology or as the social sciences. This display would be suggestive later for even clever students when they take up the thorough study of one of these sciences. An editorial in *Nature* (28.7.51, p. 130) discussed why few firms can attribute their commercial success to their research departments. The diagnosis was a lack of integrated approach, a false cleavage between science, technology and economics: also a lack of receptivity to speculative ideas. The Percy Report (1945) was quoted as attributing all this " in part to a failure of education ", and a more recent authoritative opinion is that it is " a major problem for British education ". But this aspect of what we should aim at, in mathematics in particular, seems educationally scarcely ripe for discussion, We are muddling through, largely on general impressions. Scientists and industrialists do not expect more from the schools mathematically; where really different mathematics is required from what is customary it is taken for granted that its study must be *ad hoc*; or, as long as students can mug along, side-stepping easy relevant mathematics, it is taken for granted that there is no option to this.

The third remark (A'), also in 3.1, seems complementary to the first (A) in urging the importance of going slow with calculus, or rather in being ready to turn aside to sketch curves; for these illustrate the new ideas in the calculus and help to ensure that they are assimilated. This piece of advice goes much deeper than Principle A, and one would have liked to see it made the basis of a felt need for the Calculus method, which would then appear as a tool by which might be cleared up some uncertainties about curves plotted from calculated values in order to depict related changes. In fact this *contemplation of function* seems to be the clue to the needed teaching of mathematics in schools, not as primarily a toolbox, as still often taught and examined, but with the leading aim to give pupils as clear and full a view as possible of types of controlled variations as expressed in equations. That is the answer to our *Why?* What is new here is, if anything, the emphasis: it is clear enough in the Jeffery Report, but somewhat obscured in the syllabus accompanying it. In present practice, emphasis is still distinctly on manipulations as ends in themselves. In the common tradition for many pupils, techniques are more important than appreciation. It may be added that, when we have got

further from the old outlook on pupils' needs, there will probably be something to say of the need to appreciate random variation, of interconnections, and of ramifications of relations; but for the present a better understanding of functions seems a sufficient objective; the skills or techniques involved should be clearly subsidiary to this, or left to the specialist.

Now, considering graphs more closely, for the very reasons that make them so valuable educationally, it is quite easy to get bogged in curve drawing: the interest in checking up simple phenomena like loops and cusps may lead to overlooking the broad facts of variation. There is an interesting example of this failure to delight in " function " in the Calculus Report, and it may be useful to consider it. It is not obvious why on page 55 merely a detail of the curve given by

$$x = cs^2 = \tfrac{1}{4}(c - c_3), \qquad y = s(1 + c^2) = \tfrac{1}{4}(5s + s_3),$$

where $c \equiv \cos t$, $s \equiv \sin t$, $c_r \equiv \cos rt$, etc., is dealt with. The example occurs in at least one textbook, and there the length of the arc found in the Report is also given. But in neither is any notice taken of what is of interest to those who have watched an oscillograph, viz. that the cusp arises from combining two smooth curves. (There are no cusps with Lissajous curves.) How much more interesting had the Report illustrated Principle A′ by giving the space taken up by its conventional Fig. 43 (cf. Fig. 33) to display the curve in its completeness! Even the sketching of the component functions $x(t)$, $y(t)$ might be testing for some pupils. It is of some interest too that the cusp is not at the origin, or easily moveable thither. Then the calculation of positions near the cusp may be revealing as to the power of tables of different accuracies to discriminate between neighbouring points. (Hence light won on " neighbourhood "?) There are also the symmetries, and the order in which the quadrants are occupied as t increases. From the graph too it can be seen that the algebraic equation for the curve is of 6th degree at least, as the symmetries suggest: this fact might lead some of the best pupils to some purposeful manipulation. Another question: Is there a graphical interpretation of $x/c + y/s = 2$, obtained from the original equations? A next question might naturally be: What about $x = c^2s^2$, $y = c^2(1 + s^2)$, and why is the corresponding curve less interesting? I do not know if pupils nowadays set themselves such questions.

In general what is suggested here implies a move towards independence, a greater readiness of ordinary pupils to be critical of their own work, instead of depending on authoritative answers. The need for graphical or numerical checking may be illustrated from Note 2263 (*M. Gaz.*, XXXV. 43) where, in connection with $u \equiv 2ar/z^2$, where $z^3 = r^3 + 2r^4a^4 \cos 4\theta + a^8$, it is asked why maxima and minima of u on $x^4 + y^4 = a^4$ are obtained from $du/d\theta = 0$ but not from $du/dr = 0$. Evidently, for r given, the u surface rises and falls in a modified harmonic way as θ increases, so that there are lines of minima along $\theta = \frac{1}{2}k\pi$ and ridges along $\frac{1}{4}(2k + 1)\pi$, k integral. The modification is extreme when $r = a$, and the surface thus forms infinitely high " chimneys " at

$$(a, \tfrac{1}{4}(2k + 1)\pi).$$

Simple differentiation w.r.t. r shows that the surface enters the origin $r = 0$ at a gradient $2/a$, and that $\partial u/\partial r \to 0$ as $r \to \infty$. Most schoolboys know the cylinders $x^n + y^n = a^n$ for $n = 1$, 2 and it is hoped, a few more values of n, and so the 3-dimensional curve in question can be seen passing through a minimax on the u-surface at $(a, 0, 2\frac{1}{2})$ and crossing a ridge at $(2\frac{1}{2}a, \frac{1}{4}\pi, 2^{5/4})$, as obtained by analysis from $du/d\theta$. That no result came from du/dr is due to $\cos 4\theta = -3 + 4a^4/r^4$ implying that r is inside the range $a(1; 2\frac{1}{2})$ and the $u(r)$ formula obtained is correct only within that range. (It is a good exercise to

check this last statement.) Of course where the cylinder intersects the ridge the value of u is not a maximum on the u-surface.

5. *A New Perspective.*

Granted the desirability of dominant stress on the idea of function, doubt will arise as to its feasibility. There are difficulties in plenty, and it may be of use to indicate lines which such new teaching may take. We need not wait over transition problems. The history of the teaching of graphs may suggest how far changes are needed. After the arithmetical treatment of temperature charts, etc., graphs were presented as an alternative manipulation, a way of solving simple equations. And so graph-paper was essential; this was troublesome, and in fact comparatively few pupils felt the thrill of utter correspondence of results. The emphasis was disciplinary as in the old teaching of algebra. It is difficult to see why stress has not recently been laid on beginning with sketches. Untidiness is a grim reality, but when children early are given disciplined freedom in geography and in art, it seems far from hopeless to get pictures of families of curves constructed from calculations which need not be unduly extensive. Contours present no difficulty and sets of parallel lines, and even pencils of rays, would be a welcome widening of the idea of a map. Incidentally would come practice in securing reasonable accuracy which would have a purpose obvious to the pupil; effective selection too of the values most easily calculated would be one of the most worthy skills. All this may offer one of the best hopes of solving the problem of reliable calculation, the failure to secure which has been so long and so generally deplored. This suggestion has a wide application: some time ago, when proposals such as follow were presented to a professor who had been a senior wrangler, his first reaction was "Well, this would certainly teach them arithmetic". What he had in mind was the struggle to get his able students to bring their theories to the test of calculation. Again, for the biologists, how many pupils who quote the formula $y = kx^n/(1+kx^n)$ in connection with a suggestive diagram in physiology and speculate whether n should be 1, 2, 3, 4, ... or 2·6 realise just how the corresponding family of curves behaves? A more relevant training in mathematics would have enabled them to consider the adequacy of the hypothesis behind the formula, and generally to deepen and facilitate their thought.

In what follows it should be fairly clear that it is easy to separate elementary ideas from more advanced developments, and so the suggestions have relevance to education at various stages and of different types. Also, of course, (*A*) it is inadvisable to pause till all pupils have "mastered" the simpler families. The next step after the straight contours should not be to second degree curves, save for a passing reference and a promise to return after ideas have been broadened; quadratics are dull if important. A challenge, however, could be left with certain pupils to find exceptions if they can to the four types of $y = a + bx$ according to the signs of a, b, checking from $(0, a)$, $(-a/b, 0)$; and similarly for $y = a + bx + cx^2$ according to the signs of a, c. The more the abler pupils find out for themselves, the better, though few are likely to discover the significance of $b + 2cx$ here; but plenty of talking points have been given even to the less able. In class attention should next be concentrated on cubics. With them, more general ideas can be effectively presented and a vista opened on the plotting of polynomial expressions in general. Here it seems advisable to remark that the calculation of values of $a + bx + cx^2 + dx^3$ is most effectively done at this stage from the factorial form,

$$a + x(b + x(c + xd)).$$

This gives a simple algorithm which can be extended to polynomials of higher

degree. I am puzzled as to whether the ignoring of the help this device gives towards a general view of polynomials is due to devotion to the type of question common in examinations, or to apparent difficulties of calculation, or to some more recondite difficulty. It may be mentioned that in a Distinction paper of 1945 the iterative theory of this algorithm for computation was set, but there was no suggestion of the simple form in brackets and of the use to which this expression might be put—a striking example of contentment with algebraic manipulation? It is of interest to compare the treatment of these cubics set out by Prof. A. Lodge in the *Gazette* (XI. 322) in 1919. It is almost amusing to find in an elementary textbook in use now the example

$$\tfrac{1}{2}(x+\tfrac{1}{2}(x+\tfrac{1}{2}))$$

for simplification, presumably $xa+xa^2+a^3$ when $a=\frac{1}{2}$? If so, it is sad that the important point has been so badly missed. This example may be a supreme first year test on the use of brackets, but the form it represents should be brought into routine use quite early, whatever be the skill in dealing with brackets in general, and its worth proved in practice. That stage past, there may come as need arises elaboration of the use of brackets. This now comes so early that no meaning save discipline is suggested. The experience of parallel calculations under this one factorial form, all with a purpose, could be an encouragement not to fear work with higher degree curves, and even a stimulus to explore wider possibilities in connection with factorials (e.g. synthetic division), which also are taking on a new importance. For the abler pupils a way might be opened to investigate the properties of series which would be a valuable preparation for critical appreciation of convergence later, and even the less able may be helped to some idea of why convergence is important.

A question may arise as to the presentation by the pupils of their results from many evaluations of the same expression—so unlike the present meaningless exercises of making the same substitutions in a multitude of different functions! The sketches already referred to would in many cases suffice, but for full understanding it were well that really accurate drawings be attempted in a few cases. If these are to be as efficient as possible, good use must be made of the paper available: for this preliminary sketches are essential—unless the pupil is treated simply as a child and given full instructions what to do. This procedure is in present practice generally premature judging from the difficulty in achieving real accuracy even at Certificate tests when the pupil is on his own. From examiners' remarks it would seem that there is now little satisfaction gained by pupils in pushing their drawing tools to the utmost—the joys and surprises and excitement of "all out" are confined for most pupils to games? It should be remarked that the solution of transcendental equations, where it would be reckoned that careful graphs would pay, is really carried out by the use of tables after hints from graphs. Yet actually seeing how closely the results of calculations, either pure or involving the use of tables, can be verified by eye and hand may be really thrilling; the difficulty of it all has been referred to in commenting on the cusp example in the Calculus Report. More enlightening if this careful drawing leads on to doubts as to the reliability of even fine tools, especially graph paper. Here again may be part of the solution of the problem of accurate calculation: it all involves working with a purpose; if the computing is faulty that becomes evident without any checking from the teacher.

It may be noted that even in the Jeffrey Report and some syllabuses derived therefrom positive and negative indices are jumbled together into $\Sigma_{-2}^{3}a_r x^r$, undue difficulty for the pupil being avoided by restricting the number

of non-zero coefficients to three. It is certainly interesting to mix positive and negative indices, and in a sense it is necessary to do so for a complete treatment; but this opens no immediate vista and indeed avoids generalisations which are essential in the modern use of mathematics. Further, important curves like $y=(1+x^2)^{-1}$, $y=x(1+x^2)^{-1}$ are not included.

Even more important than polynomial curves (if comparisons may be made) is the family $y=x^n$. Curves for $n=-1, \frac{1}{2}, 1, 2, 3$ are commonly asked, and even $n=3/2$; but these curves with any fractional or negative value for n are used in physics, biology, economics, etc., and the family of curves should be seen as a whole. I have heard a very able lecturer in physiology get badly confused as he was not clear about the simple relation between $y=x^n$ and $y=x^{1/n}$, and there are other more subtle, but important, phenomena here which we need not wait over now. Every schoolboy too should know the relations between $y=x^n$ in the first quadrant and $\log y=n \log x$ in all four quadrants. This work in its fulness would generally be taken later than that on polynomials.

Discussion of polynomials would naturally have led to the calculus as an aid to the sketching and drawing of curves. Here it is important, though it is not mentioned in the Calculus Report, that simplification be carried to the extreme, and in doing this the fact of limits would be pointed out repeatedly, but avoided as far as possible. This may be shocking to the expert, but after all in physics, chemistry, biology, economics interest is not in infinitesimal changes, and an approximate algebra of increments is feasible and sufficient. "Nothing to unlearn?" Certainly avoid misconceptions and waste, as the Report on the Calculus strives to point out, but really few reach the blissful (?) state where they have nothing to unlearn! It is to be pointed out bluntly to pupils that this simplification is not an adequate substitute for mathematicians to work with, but that they have to travel a long road in the clarification of ideas before they are satisfied. There may be more harm in giving the impression that the final statements in the Calculus Report are fully adequate than in putting real mathematicians on a pinnacle in their understanding of "limit" that few can hope to scale. It is important to make clear that mathematicians achieve their skill and insight largely through trying to make statements that are true (without exception). As long as we are frank the risks of oversimplification here are not serious. A reference to what is regarded as advanced economics may illustrate the advantage of definitely postponing limit theory till ideas of function are enriched, and also the likelihood of great help from well-adjusted mathematical curricula. In the Cowles Commission Monograph, "Activity Analysis of Production and Allocation" (1951) it is pointed out (p. 6) that "linearity" relates to proportionality and additivity merely, and that "polyhedral production functions derived from a finite basis ... constitute a gain in realism in many applications".

According to the Calculus Report undue attention seems to be given to notation. It is of course of the utmost importance to avoid causes of confusion, but it is of great importance too to prepare pupils for other ways of looking at many-sided truth. They will meet many instances in applied science, and even in dictionaries, of attempts to standardise notation, and the important thing is to make prominent beyond any possibility of doubt the symbolism in use, and to be prepared for its inadequacy in particular cases. In fact, I would say that the solution lies largely in the habit of tabulation, and in the construction of a relevant glossary; but details cannot be dealt with here. An example too may be noted where the increment notation is advantageous. A friend who is an able mathematician drew my attention to the "proof" in Ball's *Mathematical Recreations* that every ellipse is a circle, and remarked

that the fallacy was doubtless one of range (like one in Note 2263), but to say so would tend to confuse. Actually from $r = a + ex$ we certainly get $dr/dx = e$. Ball avoids saying that this is constant unless x is at the endpoint of its range. He merely says dr/dx is never zero, which is false. The position is clearer in terms of increments :

in a circle, $r = a$, $\delta r = 0$ whatever ($\delta\theta$ or) δx ;
in an ellipse $r = a + ex$, $\delta r = 0$ when $\delta x = 0$, and in general $\delta r = e \,.\, \delta x$.

It seems unnecessary to do more than mention here the desirability in the perspective of functions of giving greater prominence to polar graphs, nomograms (which are closely related to contours), finite differences and statistics, for their increasing use is obvious enough. But something should be said of the use of the slide rule in view of the resistance which its advocacy has met (spite of Spens Report, p. 239). A book on medical statistics (1952) goes out of its way to remark that the slide rule is convenient but not essential : " more important are tables of logarithms, squares and square roots ". A friend to whom I showed this remark bubbled over with impatience : " I never bother with tables. I use the slide rule for everything." Tables of logarithms stand rather apart, but it may be of use one day to define more clearly the efficiency of different means of calculation according to the requirements of the situation. The curious opposition to introducing the slide rule freely in schools may be due very largely to supposed expense, for it is not generally known to local authorities that satisfactory slide rule work could be done by having 12-inch rulers suitably stamped with scales for both direct and inverse slides. The view of the " practical " man that the slide rule can be learned in half an hour is rather callous, and then one finds that he does not know the illuminating fact that the slide can be used inverted, certainly he has no idea why this should be so, and such ideas the schools could make clear. Functional scales are so common that people should not be left with unnecessarily hazy ideas as to how they arise. An obvious objection for some is to the eye-strain involved, and so it is well to keep to the 10-inch rule, and make the graduations open. But eyes are better cared for than they used to be, and I have known some of the weak-eyed ones good at calculations and able to make sufficiently accurate direct estimations. In any case, it is not right to make the majority suffer because of the few who may be handicapped. Rather curiously it is said by some practical men that introduction of the slide rule will lessen the power of accurate mental calculation which is now so low. This may be considered carefully, but the main fact of course is that rough mental calculation and getting more accurate results from the scales are complementary operations, and use of the slide rule would stimulate to quick and sufficiently accurate mental work. In particular the placing of the decimal point is one of the best elementary tests of the commonsense which is required by the computer, mental or otherwise. It need scarcely be repeated that direct striving for accuracy may not be the best way to secure it, and that interest in accuracy comes when there is a ready means of checking, such as the slide rule often gives. Still more curious objection comes from the expert statistician that the slide rule does not add as his machines do. This throws interesting light on modern computation ; but the education of ordinary pupils in mathematics should not be dominated by such experts, or by pure mathematicians—though I have known a number-theorist purchase a cylindrical slide rule in the hope that it would help him in his searches. In general, the slide rule should lighten the burden of teaching, and make possible for individuals effective dealing with large numbers which now are taken to involve laborious teamwork.

6. *Filling in the Perspective.*

The foregoing are but a sample of ideas that would arise from a proper sur-

vey of efficient uses of elementary mathematics. Actually at present none really know what mathematics in schools should or might be, and it appears as if the Mathematical Association alone could take effective steps to clear the obscurity. After the Second World War, concern was shown, by electrical engineers especially, about the ineffectiveness of education. The series of reports produced were in their mathematical aspect rather conventional, yet young engineers are in many cases making great efforts to teach themselves the new types of mathematics they need. The pages of the *Gazette* are up-to-date, but they reflect these needs only in minor degree though there is much of quite general interest to discuss. An elementary lecture given recently to a Conference of Architects and Engineers on the idea of Relaxation Methods did not stir recollections of anything in the *Gazette* recently quite like that, though iteration is now widely used : it would have made the lecturer's task easier if he felt that he could have referred to iteration. Of course, such topics require consideration by experts, but the need is so general that it seems insufficient to leave it more or less to chance that there may be a brilliant individual interested who may (be made to) feel that the subject should be expounded. Even so the topic usually needs examination at different levels, and there should be others (mathematicians) near enough to the ideas involved to carry on discussion in a wider way than is possible in an expert article in the *Gazette*.

We need in the Association a number of permanent committees, which would replace those which have prepared subject reports, in order that a watch may be kept on the uses of elementary mathematics (or mathematics that may be made elementary) in journals and books. These uses are changing obviously in quantity and also in quality. It is merciful and fundamental that the non-academic demands of life be put in the forefront ; then questions of logic and outlook will have a better chance of effective treatment when dealt with in this setting. A simple division of the tasks would be between three committees concerned respectively with applications to the " exact " sciences and technology, to " life " sciences including medicine, and to social sciences including psychology. Even so the burden may be rather heavy for one committee, but the special value of using mathematics (B′) to coordinate the early teaching within these groups makes it important to take particular care to preserve so much unity. There might well be a fourth committee to attend to such coordination as is required between the groups and also to concern itself with the purely mathematical possibilities. There will be no lack of suggestions on such matters once the contrast between what pupils need and what they receive makes it clear that such a setup is needed, and in what follows we have to think of special forms of the need which might thus be met. Of course, it would be desirable to have non-mathematicians coopted to such committees : a ready response is probable, especially if the outside experts were asked to deal with limited tasks which they have helped to define.

A simple example is the article on " Statistics and Agriculture " in the *Gazette* (XXXI. 21). Such would be an opportunity for members of the biology committee to follow up with suggestions and elucidations of details which would be of help to teachers who may have an opportunity to use some aspect of the procedure described. Actually this is now being done for fractional replication by the Industrial Applications Section of the Royal Statistical Society with facilities for publicity rather less than has the Mathematical Association.* (Note the transgression of the natural boundaries suggested

* Since this was written a new quarterly, *Applied Statistics*, has appeared and has had a welcome beyond expectations—fresh evidence of the extent of the need emphasised here.

above for the groups of associated studies: this is specially likely to occur with Statistics.)

Plans for mathematical teaching in economics have quite a long history but they seem to have had no influence on the schools. Now there appears to be a definite tendency, which must be disastrous, to dispense with mathematics and trust to suggestion. Comment is difficult. On the other hand, at a Conference on Automatic Control in 1951 it was suggested by an engineer that teamwork was needed between economists, engineers and mathematicians with a view to providing a physical analogue of the macro-economic system sufficiently comprehensive and accurate to solve certain of the problems of economics. Again, as to another possible task for the sociology group, a score of years ago a few mathematicians made a serious effort to cope with the mathematical problems in psychological research; now factor analysts largely go their own way and complain of lack of cooperation of mathematicians, even suggesting that mathematics has not developed far enough to cope with the problems of the analysts. This is a situation that concerns teachers most intimately. No one would say that the problems connected with selection at any stage will yield to the attack of a standing committee, but it could probably define the issues more satisfactorily than they are now understood and, if functioning humbly, would not be a hindrance to the Newton we require here, when he does come along.

The activities and interests of these permanent committees would be very varied. Links would be established with other bodies, but it may be expecting too much that they intervene in situations such as that described in *Nature* recently (28.6.52, p. 1880): "Geologists have been at fault in allowing the mathematicians to develop the idea of isostasy in a manner that has been largely unprofitable." This may suggest complementary activity which our committees would tend to foster. A borderline case occurs in a book (1952) on Statistics by a physiologist and a pharmacologist: in a preliminary statement on procedure they affirm that the idea of causality can be dispensed with. Eddington's distinction between causation and causality (*N.P.W.* 285) is apparently not in view, and it seems confusing for future doctors that they are to look for associations, not for causes! Things need clarifying: we mathematicians have apparently ceased to teach logic, even the status of necessary and sufficient conditions, and it should be settled whose task this is. In the present case the question of "cause" is dealt with faithfully in the Croomian lecture (*Lancet* 5.7.52, p. 1). "A thorough course in Physics is the best possible preparation for Law" quotes Millikan in his *Autobiography* (p. 140). It may not be a completely satisfactory ideal, but we are slack in some way if the same cannot be said of Mathematics where logical situations are more varied and should be made clear. Help might be given too in cases such as where a distinguished biochemist half in jest regretted that he had ventured to be fashionable in adopting tracer technique: the result was that he became involved with a mass of data, embodied in curves, which he found hard to interpret! The function of mathematics is generally critical rather than constructive, but even so it might be of aid in the selection of concepts for measurement.

More certainly would the exact sciences committee have to evaluate the situation described by Kron in *Equivalent Circuits of Electrical Machinery* (1951): "The absence of mathematics in the establishment of the equivalent circuits should not be construed as an attempt to belittle the importance of a strictly mathematical development of equivalent circuits. As a matter of fact, practically all equivalent circuits given have originally been developed mathematically first, and only after the availability of many examples did the organic unity of all circuits become apparent. With unusual windings . . . the

avoidance of mathematics may become quite a nuisance, and in any doubtful case a mathematical set of equations must serve as the final court of authority for the correctness of the model." The educational significance of this should be carefully considered. It may be the best attitude that is available in certain circumstances. In justifying the procedure he adopted in *Statistical Methods for Research Workers*, which has run to a dozen editions since 1925, Sir Ronald Fisher wrote "The practical application of general theorems is a different art from their establishment by mathematical proof. It requires fully as deep an understanding of their meaning, and is, moreover, useful to many to whom the other is unnecessary". This has similar educational implications, and it would be distinctly interesting to have committees of different texture consider just what these are.

The work of these committees may seem formidable, but actually a great deal could be done by correspondence and by informal meetings of individuals with the same interests. This would be facilitated if it were agreed that publicity be left mainly to individuals : in this way more of the debate would be heard by the public. If the pace of change continues as now, there will be little need for authoritative statements. It is clear that the burden of the committees will fall, especially to begin with, on college teachers : the struggle to eke out salaries makes it simply impossible for teachers in schools to play now the part they should. There is the further problem, the most urgent now, of defining the mathematical tasks of the Secondary Modern Schools. This is being blanketed at present by the fact that it is felt that whenever the chance occurs, it is only fair that pupils, even if few, should be prepared for the General Certificate. (Parents and employers, it may be noted, share the responsibility here for effort largely misdirected.) College teachers may feel that they cannot be very helpful in this respect, but the first task is to clear the way to most worthwhile lines of experiment, and they can at least make it nearly certain that all possibilities have been considered.

7. *Opportunities in Secondary Modern Schools.*

Some specially enterprising or gifted teachers in S.M. schools have turned away from the ordinary curriculum to matters more directly interesting and useful, and there is of course supreme value in getting pupils thus to be active. There are, however, other possibilities, no more difficult than the surveying and astronomy often advocated, that lead to wider vistas in which world activities can be seen more as a whole. The emphasis on economics and medicine in the preceding sections may suggest possibilities for the general run of pupils that have not yet been explored. It is possible too to consider at the moment the quality control chart as the touchstone of mathematical work in S.M. schools. It is a device readily appreciated by ordinary people in industry. In school the approach to it would be much broader. The idea of frequency distribution is almost obvious and should be well worked over. Simple tables of random numbers could be tested and used comparatively early. The treatment of course would not be continuous but would appear as arithmetic in various forms. More exciting than the spinning of coins would be the tossing of the tops from milk bottles : the usual team work would give the effect of increasing sample size ; then to consider the bias tops could have sectors of different sizes cut from them before being tossed. And so with homogeneity, method of tossing, etc. Whether the teaching could or should be used to show the folly, futility and poor fun for most fillers-in of football coupons may be disputed, but hints as to the broader aspects of the working of chance cannot but be beneficial. In this connection, simple questions of design of experiment may be raised—after all Sir Ronald Fisher begins his second path-breaking book, *The Design of Experiments* (1935), with the tasting of

milk in tea-cups. In these days such questions need not be limited to country schools—as in the experience described by Sir John Russell in the Spens Report (p. 190). Some teachers have emphasised individual work projects even at the cost of systematic instruction. Mathematical "literacy" needs to be defined, but a pupil who has worked over a set of ideas taken from the foregoing should be able to use them to advantage in several sets of conditions after leaving school.

In all this opportunities will be taken to stir up interest in number by showing patterns in arithmetic, beginning with the 9s remainders in multiplication tables, series of powers, etc. For pupils who have special difficulty in manipulating symbols an outlet may be found in the combination of simple waveforms which are part of our everyday experience, and even of our jargon, e.g., frequency band, square wave pulse, etc. used rather loosely. Thus may come confidence in symbols, e.g. in testing whether a square wave is a particular case of a rectangular wave. As skill of hand increases, practice may be given in geometrical exercises to test the correctness of geometrical propositions, especially those encountered in use. Quite obscure properties of which there would be no likelihood of finding a proof may thus with advantage be verified, e.g. that used in the design of alternating electrical machines. The centroids of equilateral triangles described externally on the sides (a, b, c) of any triangle form an equilateral triangle of side l given by $6l^2 = 4\sqrt{3}\Delta + \Sigma a^2$, where Δ is the area of the given triangle. That may stir the sense of wonder, if not of logic.

A deep test of our teaching is whether we have imparted a feeling of unity in the pupil's outlook, a sense of wholeness. It is said that in the medical course things come into focus only towards the end, but in mathematics we deal with more pliable material and a feeling of purpose and design should be present at all stages; vistas would of course broaden as we proceed. Complementary to this is the inculcation of the flexibility of approach to a problem that increases power—not simply a teaching of *the* way to tackle a problem, but the awareness of possibly complementary ways of tackling it and the choice without fluster of that which seems appropriate. Ability to take different good viewpoints is a lesson of such value that it should be desired to become a second nature with every child—an intellectual, as well as a social, toleration, which points the way to alertness and openmindedness, and even to wisdom.

J. M

1779. The side flight is probabilistic π, quite a subject in itself: it is concerned with the making of trials of various possible conjunctions of pairs of prime numbers that are prime to each other; or the random contact of regular objects, whose probabilities are expressible as formulas that contain the number π. Thus, Chartres, in 1904, made a random selection of 250 primes and found that 154 of them were prime to each other; and as it is well known that the probability of the conjunction is $6/\pi^2$, his trial amounted to an estimate for π of 3·12.—N. T. Gridgeman, "Circumetrics", *The Scientific Monthly*, (1953), **77**, 33. [Per Mr. G. N. Copley.]

MATHEMATICAL NOTES

2392. *The nedians of a plane triangle.*

The medians of a plane triangle ABC are the straight lines AD, BE, CF drawn from the angular points to the midpoints of the opposite sides. It is well known that the medians are concurrent in G, the centroid of the triangle, and also that

$$\Sigma(\text{medians})^2 = \tfrac{3}{4}\Sigma(\text{sides of triangle})^2.$$

If the sides of the triangle are, in order, divided such that the short section of each side is $1/n$th the length of the side, and the points of subdivision are joined to the opposite angular points the joining lines may be called the

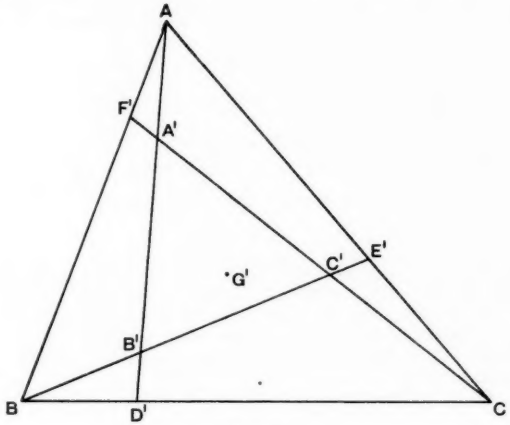


FIG. 1.

Nedians of the triangle (the name recalls the n) and the triangle formed by the nedians may be called the *nedian triangle*. Thus, in Fig. 1, where

$$BD' = BC/n, \qquad CE' = CA/n, \qquad AF' = AB/n,$$

the nedians are AD', BE', CF' and the triangle $A'B'C'$ is the nedian triangle. It may be shown that

$$\Sigma(\text{nedians})^2 : \Sigma(\text{sides of triangle})^2 = (n^2 - n + 1) : n^2.$$

Putting $n = 2$ gives the case of the medians.

It may also be shown that

$$\text{area } A'B'C' : \text{area } ABC = (n-2)^2 : (n^2 - n + 1).$$

For $n = 3$, we have the value $1/7$ for the numerical factor.*. In Figures 1 and 2, n is equal to 4.

As the division points of the sides may be taken in either the anti-clockwise direction or the clockwise direction there are two sets of nedians for any given value of n. Therefore I call the nedians AD', BE', CF' of Fig. 1 the *forward*

* The case where $n = 3$ is figured on p. 8 of Steinhaus' *Mathematical Snapshots* (Oxford University Press, 1950).

nedians and the triangle $A'B'C'$ the forward nedian triangle, and the nedians AD'', BE'', CF'' of Fig. 2, the *backward* nedians and the triangle $A''B''C''$ the backward nedian triangle.

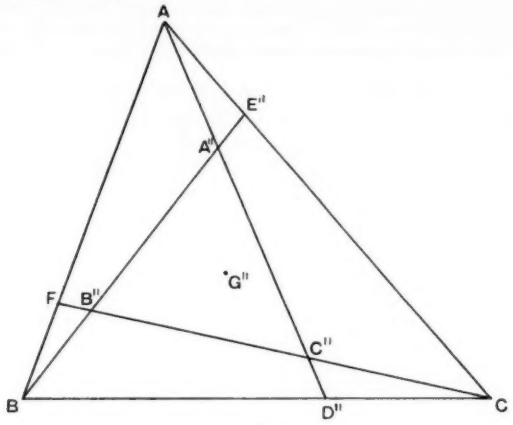


FIG. 2.

The above equations hold for both the forward and the backward nedians. Thus we have for the same original triangle and the same value of n two triangles $A'B'C'$, $A''B''C''$ of the same area but not of the same perimeter or shape. They have, however, the same centroid G, which is the centroid of the original triangle. If we insert the forward and backward nedians of the triangle for the same value of n in one diagram, (Fig. 3, with $n=5$), we observe many concurrences. That this is so follows from Ceva's theorem.

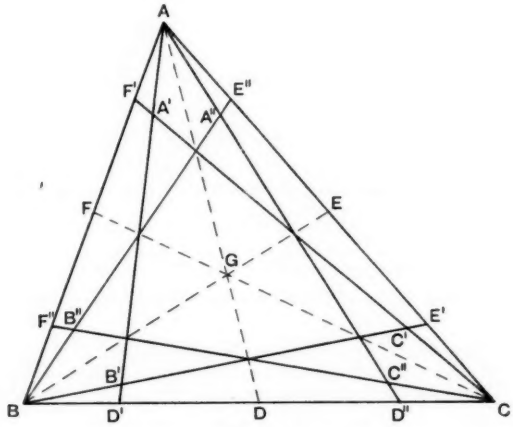


FIG. 3.

A nedian hexagon is also formed ; it is that part common to the two triangles.

Other theorems follow. For example, the area of a triangle whose sides are the medians

$$= \tfrac{3}{4} \text{ (area of the original triangle)}$$

and the area of the triangle whose sides are the nedians

$$= \{(n^2 - n + 1)/n^2\} \text{ (area of the original triangle).}$$

JOHN SATTERLY.

2393. *Indeterminate forms and l'Hospital's rule.*

This note is designed to offer a proof of l'Hospital's Rule for indeterminate forms that may be more palatable to the beginning student than the usual one based upon the Theorem of the Mean.

I. The Form $\frac{0}{0}$.

Let $f(x)$ and $g(x)$ be differentiable functions at $x = a$ such that

$$f(a) = g(a) = 0.$$

Since the ratio $f(a)/g(a)$ is meaningless, we wish to examine values of

$$\frac{f(x)}{g(x)}$$

as x approaches a. To do this, consider x as a parameter defining f as a differentiable function of g. As x varies (Fig. 1) the point (g, f) describes a curve through the origin (where $x = a$). Clearly

$$\frac{f(x)}{g(x)} = \tan\theta,$$

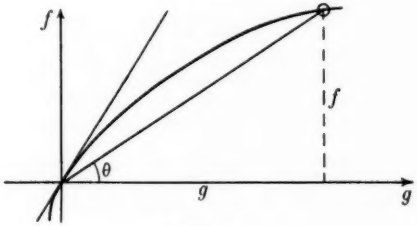


FIG. 1.

the slope of the secant, and as f and g approach zero (i.e., as $x \to a$) then $\tan\theta$ approaches the slope of the curve at the origin. That is,

$$\operatorname*{Limit}_{x \to a} \frac{f(x)}{g(x)} = \operatorname*{Limit}_{x \to a} (\tan\theta) = \left.\frac{df}{dg}\right|_{x=a}.$$

Expressing df/dg as the quotient of two differentials, we have

$$\operatorname*{Limit}_{x \to a} \frac{f(x)}{g(x)} = \left.\frac{df}{dg}\right|_{x=a} = \left.\frac{f'(x)dx}{g'(x)dx}\right|_{x=a} = \frac{f'(a)}{g'(a)}, \quad g'(a) \neq 0.$$

II. The Form $\frac{\infty}{\infty}$.

Let $f(x)/g(x)$ be such that $f(x)$ and $g(x)$ increase without bound as x approaches a. If x again be considered a parameter defining f as a function of g, differentiable as $f\to\infty$, $g\to\infty$, the graph, Fig. 2, of the function extends to infinity. We assume that $\underset{x\to a}{\text{Limit}}\,[f'(x)/g'(x)]$ exists.

The problem is reduced to the form 0/0 of I by effecting a *geometric inversion* with respect to the origin. If $p:(g,f)$ and $P:(G,F)$ be inverse points, then as $f, g\to\infty$, $F, G\to 0$.

The tangent t at any point p of the original curve has slope df/dg and

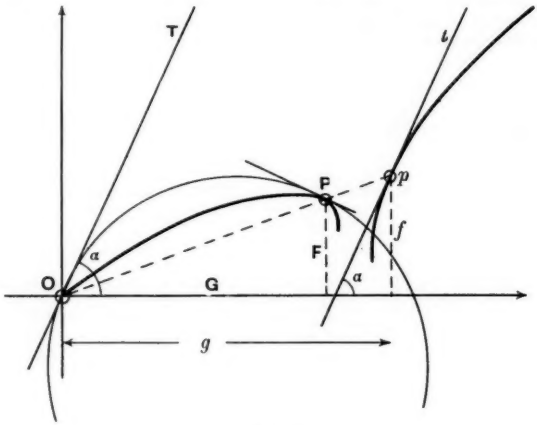


FIG. 2.

inclination α. This tangent inverts into a circle through P and the origin. The g-axis inverts into itself. Since under inversion angles are preserved in magnitude, the circle is tangent to the inverse curve at P and has the tangent T at the origin parallel to t. Now f'/g' has a limit; thus there is a limiting direction for T and t.

As $p\to\infty$, then $P\to 0$, and the tangent common to both circle and inverse curve at P approaches the limiting position of T. Its limiting direction, since $P:(G,F)$ belongs to both circle and inverse curve, is

$$\underset{P\to 0}{\text{Limit}}\,\frac{F}{G}.$$

Because T is always parallel to t, their slopes are always equal and equal in the limit. Accordingly

$$\underset{P\to 0}{\text{Limit}}\,(F/G)=\underset{p\to\infty}{\text{Limit}}\,(f'/g').$$

But at all points $F/G=f/g$, and thus

$$\boxed{\underset{p\to\infty}{\text{Limit}}\,\frac{f(x)}{g(x)}=\underset{p\to\infty}{\text{Limit}}\,\frac{f'(x)}{g'(x)}}$$

ROBERT C. YATES.

2394. *The associated Legendre polynomial.*

Very little reference is given in normal textbooks to the recurrence and integral relations involving the associated Legendre polynomial of the first kind, P_n^s, given by

$$P_n^s(\mu) = (1-\mu^2)^{s/2}\,\frac{d^s}{d\mu^s}P_n(\mu) = \frac{(1-\mu^2)^{s/2}}{2^n \cdot n!}\,\frac{d^{n+s}}{d\mu^{n+s}}(\mu^2-1)^n.$$

The following is a list of such relations that I have found useful. Obviously they are not all independent.

(*a*) $\sqrt{(1-\mu^2)} \cdot P_n^s = (P_{n+1}^{s+1} - P_{n-1}^{s+1})/(2n+1)$;

(*b*) $(2n+3)\mu P_{n+1}^s = (n+s+1)P_n^s + (n-s+2)P_{n+2}^s$;

(*c*) $2s\mu P_{n+1}^s = \sqrt{(1-\mu^2)} \cdot P_{n+1}^{s+1} + (n+s+1)(n-s+2) \cdot \sqrt{(1-\mu^2)} \cdot P_{n+1}^{s-1}$;

(*d*) $(n+s+1)P_n^s = (n-s+1)\mu P_{n+1}^s + \sqrt{(1-\mu^2)} \cdot P_{n+1}^{s+1}$;

(*e*) $(1-\mu^2) \cdot dP_{n+1}^s/d\mu = (n+2)\mu P_{n+1}^s - (n-s+2)P_{n+2}^s$;

(*f*) $\displaystyle\int_{-1}^{1} P_n^s P_m^s\,d\mu = 0, \quad n \neq m,$

$$= \frac{(n+s)!}{(n-s)!} \cdot \frac{2}{2n+1}, \quad n = m;$$

(*g*) $\displaystyle\int_{-1}^{1} P_n^s P_n^t\,d\mu = (-1)^{(s+t+s)/2} \cdot \frac{(n+t)!}{(n-s)!} \cdot \frac{2}{2n+1}, \quad s+t \text{ even}, \ s \geqslant t;$

(*h*) $\displaystyle\int_{-1}^{1} P_n^{s+1} P_m^{s-1}\,d\mu = 0, \quad m > n, \quad \text{or} \quad n-m \text{ odd},$

$$= -\frac{(n+s-1)!}{(n-s-1)!} \cdot \frac{2}{2n+1}, \quad n = m,$$

$$= \frac{4s\,(n+s-3)!}{(n-s-1)!}, \quad m = n-2;$$

(*i*) $\displaystyle\int_{-1}^{1} \mu P_n^s P_m^s\,d\mu = 0, \qquad m \neq n \pm 1,$

$$= \frac{(n+s+1)!}{(n-s)!} \cdot \frac{2}{(2n+1)(2n+3)}, \quad m = n+1;$$

(*j*) $\displaystyle\int_{-1}^{1} (1-\mu^2)^{-\frac{1}{2}} \cdot P_n^s P_{n+1}^{s+1}\,d\mu = 2(n+s)!/(n-s)!$;

(*k*) $\displaystyle\int_{-1}^{1} (1-\mu^2)^{-\frac{1}{2}} \cdot P_n^s P_{n+1}^{s-1}\,d\mu = 0$;

(*l*) $\displaystyle\int_{-1}^{1} \sqrt{(1-\mu^2)} \cdot P_n^s P_m^{s+1}\,d\mu = 0, \quad m \neq n \pm 1,$

$$= \frac{(n+s+2)!}{(n-s)!} \cdot \frac{2}{(2n+1)(2n+3)}, \quad m = n+1;$$

(*m*) $\displaystyle\int_{-1}^{1} (1-\mu^2)^{-1} \cdot P_{n+1}^s P_{m+1}^s\,d\mu = \frac{(n+s-1)!}{s\,(n-s-1)!}$ if $m = n-2,$

$$= \frac{(n+s+1)!}{s\,(n-s+1)!}$$ if $m = n$;

(*n*) $\int_{-1}^{1}(1-\mu^2)^{-1}P_n^s P_n^t\,d\mu=0, \quad s\neq t;$

(*o*) Writing D for $d/d\mu$,

$$\int_{-1}^{1}(1-\mu^2)\,.\,DP_n^s\,.\,DP_m^s\,d\mu=\int_{-1}^{1}\{m(m+1)-s^2/(1-\mu^2)\}P_n^s P_m^s\,d\mu.$$

The following is a useful method for proving some of the integral relations. Consider for example (*g*) above. Writing D for $d/d\mu$, and using the definition of P_n^s, we obtain

$$I=\int_{-1}^{1}P_n^s P_n^t\,d\mu$$
$$=\int_{-1}^{1}\frac{(-1)^{(s+t)/2}(\mu^2-1)^{(s+t)/2}}{2^{2n}\,(n!)^2}D^{n+s}(\mu^2-1)^n\,.\,D^{n+t}(\mu^2-1)^n\,d\mu.$$

Integrating this by parts $n+t$ times, since $s>t$,

$$I=\frac{(-1)^{(n+t)+(s+t)/2}}{2^{2n}\,(n!)^2}\int_{-1}^{1}(\mu^2-1)^n D^{n+t}\{(\mu^2-1)^{(s+t)/2}D^{n+s}(\mu^2-1)^n\}d\mu.$$

Since now $s+t$ is even,

$$I=\frac{(-1)^{(2n+3t+s)/2}}{2^{2n}(n!)^2}\int_{-1}^{1}(\mu^2-1)^n D^{n+t}\left\{\frac{(2n)!}{(n-s)!}\mu^{n+t}+\ldots\right\}d\mu$$
$$=\frac{(-1)^{(2n+3t+s)/2}}{2^{2n}(n!)^2}\int_{-1}^{1}\frac{(2n)!\,(n+t)!}{(n-s)!}(\mu^2-1)^n\,d\mu$$
$$=(-1)^{(3t+s)/2}\,.\,\frac{(n+t)!}{(n-s)!}\,.\,\frac{2}{2n+1}\,.$$

G. POWER

2395. *Note on Dr. Maxwell's article : " Some properties of the nine-points circle ".*

1. In Dr. Maxwell's elegant article, " Some properties of the nine-points circle " (*Gazette*, XXXI (1947), pp. 266–9), he expresses symbolically the equations of many of the lines and circles connected with a triangle ABC in terms of four fundamental ones. He was, however, unable so to express (i) the medians ; (ii) the circles such as ADB, ADC (where D is the midpoint of BC) ; (iii) the Euler line. As Dr. Maxwell remarks, any one of these determines the others.

This note shows how the problem may be solved. The method used is, unhappily, not at all elegant, but perhaps it is justified by the interesting results.

2. If P, Q, R are the feet of the altitudes, we take rectangular cartesian coordinates with BPC, PA as axes. Let the coordinates of A, B, C be $(0, p)$, $(-r, 0)$, $(q, 0)$ respectively. Then it is easily shown that the equations of the circles $BCQR$ (α'), $CARP$ (β'), $ABPQ$ (γ') and the nine-points circle (δ) are given by

$$\alpha' \equiv x^2+y^2+(r-q)x-qr=0\,;$$
$$\beta' \equiv x^2+y^2-qx-py=0\,;$$
$$\gamma' \equiv x^2+y^2+rx-py=0\,;$$
$$\delta \equiv x^2+y^2+\tfrac{1}{2}(r-q)x-\tfrac{1}{2}(p+qr/p)y=0.$$

The equation of the median AD is

$$2px+(q-r)y-p(q-r)=0,$$

which is easily shown to be

$$(q-r)\alpha' - r\beta' + q\gamma' - 2(q-r)\delta = 0.$$

We now put $q/r = \lambda_2/\lambda_3$, and the corresponding ratios for the sides CA, AB equal to λ_3/λ_1, λ_1/λ_2 (the product of the three is equal to unity, by Ceva's theorem). We also use the circles $AQHR$ (α), etc., instead of α', etc., to bring our results into line with Dr. Maxwell's, using his identities $\alpha' \equiv 2\delta - \alpha$, etc. The medians AD, BE, CF are then given by

$$\begin{array}{rrr} (\lambda_3-\lambda_2)\alpha + & \lambda_3\beta - & \lambda_2\gamma + 2(\lambda_2-\lambda_3)\delta\ ; \\ -\lambda_3\alpha + (\lambda_1-\lambda_3)\beta + & & \lambda_1\gamma + 2(\lambda_3-\lambda_1)\delta\ ; \\ \lambda_2\alpha - & \lambda_1\beta + (\lambda_2-\lambda_1)\gamma + 2(\lambda_1-\lambda_2)\delta. & \end{array}$$

3. The circle ADB belongs to the pencil formed by the circle on AD as diameter ($4\delta - \beta - \gamma$) and the median AD, and also to the pencil formed by the circle ABC ($4\delta - \alpha - \beta - \gamma$) and the line AB ($2\delta - \alpha - \beta$). A simple calculation gives its symbol as

$$\tfrac{1}{2}(\lambda_3/\lambda_2 - 1)(\alpha + \beta) - \gamma + (3 - \lambda_3/\lambda_2)\delta.$$

Similarly, the circle ADC is given by

$$\tfrac{1}{2}(\lambda_2/\lambda_3 - 1)(\alpha + \gamma) - \beta + (3 - \lambda_2/\lambda_3)\delta.$$

(The multiples have been chosen to make the coefficients of x^2 and y^2 unity in each case, to conform with Dr. Maxwell's notation.)

4. The Euler line belongs to the pencil formed by BG, CG and also to the pencil formed by BH, CH. A simple calculation gives its symbol as

$$\lambda_1(\lambda_2 - \lambda_3)\alpha + \lambda_2(\lambda_3 - \lambda_1)\beta + \lambda_3(\lambda_1 - \lambda_2)\gamma.$$

E. J. F. PRIMROSE.

2396. *An impossible construction.*

This note is to show that a triangle is, unless equilateral, determined by its circumcentre, orthocentre and incentre; but that, when given these points, we cannot in general construct the triangle with ruler and compasses.

We have, in the usual notation,

$$OI^2 = R^2 - 2Rr \qquad \text{(Euler)}$$

and

$$NI = \tfrac{1}{2}R - r, \qquad \text{(Feuerbach)}$$

giving

$$OI^2 = 2R \,.\, NI.$$

Thus, R is readily constructed, and we can construct the circumcircle, the nine-point circle and the incircle. We have also

$$\begin{aligned} OI^2 &= 4NI^2 + 4r \,.\, NI \\ &> 4NI^2, \end{aligned}$$

that is,

$$OI > 2NI,$$

so that I lies within the circle on diameter GH which is the locus of points P such that $OP = 2NP$. (This circle is not without other significance: it is coaxal with the circumcircle, the nine-point circle and the self-polar circle.) Thus for there to be a triangle with the given points as centres, it is necessary for I to lie within the circle on diameter GH. This condition may be shown to be sufficient, provided that I is distinct from N, but a formal proof is rather delicate. For certain positions of I outside this circle, there is a triangle of which the given points are circumcentre, orthocentre, and an ecentre.

The data suffice to determine the conic whose foci are O, H and whose auxiliary circle is the nine-point circle; this conic touches the sides of the triangle. This conic is singular only if $OH = R$, in which case the triangle is right-angled with vertex at H and is easily constructed. Otherwise the sides of the triangle are common tangents of this conic and the incircle. It is found

that three of the four common tangents meet in pairs on the circumcircle, these forming the required triangle which is thus unique. The fourth common tangent seems to be just another addition to the many lines associated with the triangle. It is one of a set of four, each other being the fourth common tangent of the conic and an ecircle.

It remains to be shown that the triangle cannot, in general, be constructed with ruler and compasses. We can express R, r, ρ in terms of the sides a, b, c of the triangle thus :

$$16\Delta^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c),$$
$$R = abc/4\Delta,$$
$$r = 2\Delta/(a+b+c),$$
$$\rho^2 = -(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)/32\Delta^2.$$

(ρ, the self-polar radius, is imaginary for an acute-angled triangle, but we shall deal only with its square which is always real and can be constructed since $OH^2 = R^2 + 2\rho^2$.) Evaluating $a^2+b^2+c^2$, $b^2c^2+c^2a^2+a^2b^2$, $a^2b^2c^2$ in terms of R, r, ρ from these expressions, we find that a^2, b^2, c^2 are roots of

$$x^3 - 2(4R^2-\rho^2)x^2 + \{(4R^2+2r^2-\rho^2)^2 + 16Rr^3\}x - 16R^2r^2\{(2R+r)^2-\rho^2\} = 0.$$

It follows from this that $s^2 = (2R+r)^2 - \rho^2$ (a result which should admit of geometrical proof), so that the semiperimeter s can be constructed, also Δ since $\Delta = rs$.

We consider the special case in which $OI = 2$, $NI = \frac{1}{2}$ and OI, NI are at right angles. In this case $R = 4$, $r = 3/2$, $\rho^2 = \frac{1}{2}$, and a^2, b^2, c^2 are roots of

$$x^3 - 127x^2 + 4840x - 51696 = 0.$$

This equation, it may be verified, has no integer root and so no rational root. Consequently, the squares of the sides are quantities involving cubic surds which cannot be constructed with ruler and compasses, and so the general construction is impossible with these instruments.

Apart from the case of the right-angled triangle, two cases may be mentioned in which the construction is possible. If I lies on OH, the triangle is isosceles with a vertex on OH, while if NI is perpendicular to OH, the triangle has a vertex on NI at which the angle is 60°. JOHN LEECH.

2397. *Is there any connection?*

In the polarity with respect to the self-polar circle of a triangle, the orthocentre is the pole of the line at infinity and the centroid is the pole of the radical axis of the circumcircle, the nine-point circle and the self-polar circle.

In the harmonic polarity with respect to the triangle (in which the polar of a point is the line containing the harmonic conjugates of the intersections of the joins of the point to the vertices with the opposite sides with respect to the vertices on those sides), the centroid is the pole of the line at infinity while the orthocentre is the pole of the radical axis. Is this interchange significant?

The corresponding results are true of the orthocentric tetrahedron and of the orthocentric simplex in n dimensions. JOHN LEECH.

2398. *Wisden's Cricketers' Almanack and the teaching of statistics.*

With the introduction of a paper in Statistics into the Mathematics Alternative Syllabus A of the London University General Certificate of Education, many teachers will be faced with the problem of finding suitable examples as illustrative material. The usual sources (*e.g. Monthly Digest of Statistics*) are well known, but many teachers may be unaware of the interesting data and statistical exercises which can be obtained from *Wisden's Cricketers' Almanack*. This almanack, published annually by Sporting Handbooks Ltd., contains

among other items, the complete scores of all first class cricket matches played during the previous season, batting and bowling averages for first-class and minor county cricket, and a " records " section.

From this data it is possible to find examples covering practically all the topics mentioned in the syllabus. Typical examples are listed below under broad headings.

1. *The Design of Samples.*

(*a*) Take a sample of matches to determine the percentage of unfinished games in the County Championship. The percentage unfinished in all such games can be easily found directly from the Championship table, so this example can be used to illustrate the effect of increasing sample size.

(*b*) Find the professional/amateur ratio of players by sampling methods. This can be done by taking a sample of players (listed in each county's averages with the number of matches played in) or by a sample of matches.

In both these examples, care must be taken to avoid bias, *i.e.* the proportion of unfinished games is affected by the weather, while the professional/amateur ratio is affected by the increase in the number of amateurs playing in August.

2. *Frequency Distributions.*

(*a*) J-Shaped.

(i) The number of individual innings between 0–9, 10–19 ... runs for any County.

(ii) The distribution of the number of test match appearances of individual players.

(iii) The number of times individual players have completed the " Cricketer's Double ", *i.e.* scored 1000 runs and taken 100 wickets in the same season.

(*b*) Normal. The rate of scoring off individual bowlers (measured by the number of runs scored off the bowler per 100 balls bowled).

(*c*) U-Shaped. The number of County Championship matches played in by individual players.

(*d*) Skew. The completed score in the first innings of County Championship matches.

For more advanced students, the distribution of the number of test match appearances gives an excellent example of how the distribution of the sample mean approaches normality as the sample size increases. If the players are grouped alphabetically in fives and then in tens, and the three distributions of the means of samples of size 1, 5 and 10 are plotted on probability paper (or as histograms), the result is remarkable.

3. *Measures of central tendency.*

(*a*) The individual scores of a batsman for the whole season can be extracted and the mean and median calculated. The accuracy of extraction can easily be checked by comparison with the season's averages.

(*b*) Calculation of mean and median from data produced in the frequency distributions 2(*a*) (i) above. A table giving the total number of runs per wicket by each county is included in the recent annuals. This enables one to compare with the results for grouped data after the problem of what to do with the " not out " scores has been considered.

4. *Rank Correlation.*

(*a*) The rank correlation between the positions in the County Championship table in two successive years.

(*b*) From the table of the average runs per wicket for and against each county two different measures can be formed, *i.e.* the *difference* between the averages for and against, and the *ratio* of these averages. By arranging the counties in the order of *differences*, and then in order of *ratios* and calculating

the rank correlation in each case with the order in the final championship table, the two measures can be compared.

5. *Scatter Diagrams.*

(*a*) The average runs per wicket for and against for each county.

(*b*) The relationship between scores made by pairs of recognised opening batsmen.

6. *Regression.*

(*a*) The number of runs scored in an innings and the rate at which they were scored (again measured by the number of runs per 100 balls bowled).

(*b*) The relationship between mean and median for individual batsmen.

The advantages of using data from this source are :

(*a*) Undue emphasis is not laid on the normal distribution.

(*b*) The data are usually in a convenient form for extraction and plenty of cross checks are available.

(*c*) Classification is not always straightforward. Practical points such as how to treat " not out " or otherwise incompleted innings are continually cropping up.

(*d*) The task of extraction never becomes boring (especially for boys of all ages!).

Most of the items mentioned above are routine statistical calculations, but Wisden has one great advantage over the better known sources in that it can provide the data for several interesting and unsolved problems, any one of which would serve to illustrate the methods of the statistical approach to a research problem. Among them are the following :

1. Are slow left arm bowlers more difficult to score from than the other types? If the available knowledge of cricket is insufficient to classify players according to the nature of their bowling, the alternative problem " Are left arm bowlers more difficult to score from than right arm bowlers? " can be tackled as all left arm bowlers are marked as such in the season's averages.

2. Are University batsmen more successful when playing in University games than when playing County Championship matches later in the season?

3. In the fourth innings of a match a side may either score faster than the average rate in order to win in time, or score slower in an effort to play out time. This would suggest that the dispersion of the rate of scoring in the fourth innings was greater than that in the first innings. Is this so?

4. Is the rate of scoring higher in the Minor County Championship than in the County Championship?

5. Is there any change over the last 50 years in the number of Blues awarded by the two Universities to players from the major Public Schools?

6. What feats by a bowler are equivalent to

(*a*) 1000 runs in a season,

(*b*) 100 runs in an innings

by a batsman?

The knowledge of statistics required for the solution of these problems is within the scope of the syllabus mentioned in the beginning, and they provide a very good introduction to the necessity for and use of tests of significance.

G. R. LANGDALE.

2399. *On the Hessian solution of the cubic.*

If the Hessian of the cubic ax^3+bx^2+cx+d has complex linear factors $px+u \pm iv$, then the constants A and B, found by equating coefficients of x^3 and x^2 in the identity

$$ax^3+bx^2+cx+d \equiv A(px+u+iv)^3+B(px+u-iv)^3,$$

are complex conjugates. Hence we may write $(-A/B)^{1/3} = \exp i\alpha$, (where α has any of the three values $\frac{2}{3}\arg A + \pi$, $\frac{2}{3}\arg A + \pi \pm \frac{2}{3}\pi$) and the zeroes of the cubic are then given by

$$(px + u - iv)/(px + u + iv) = \exp i\alpha,$$

whence

$$px + u = iv(1 + \exp i\alpha)/(1 - \exp i\alpha) = -v \cot \tfrac{1}{2}\alpha.$$

This artifice gives a simple method of completing the solution of a cubic equation in the only case which offers any difficulty. It also shows incidentally that the cubic has distinct real roots if the Hessian has complex factors. This condition is also necessary, since if the factors of the Hessian are real and distinct then A and B are both real, and it is easily seen that this implies two complex roots of the cubic, while it is well known that if the Hessian is a perfect square or vanishes identically then the cubic has repeated roots.

This result in discrimination also follows at once from the fact that the discriminants of the cubic and Hessian are identical. The discriminant of the cubic is, however, relatively difficult to remember and to calculate, while the Hessian can be written down easily provided that it is formed by putting $y = 1$ in the Hessian

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

of the form $f(x, y) \equiv ax^3 + bx^2y + cxy^2 + dy^3$ and not by substitution in the unmemorable quadratic which the Hessian is defined to be by many authors, who thereby deprive this method of solution of most of its value.

D. A. T. WALLACE.

2400. *Calculation of curvature and torsion of a twisted curve.*

The following procedure allows the curvature and torsion of a twisted curve to be determined independently of each other, and with relatively little calculation.

Taking θ as the parameter, and using dots and dashes to denote differentiation with respect to θ and the arc length s respectively, we have, using the notation and sign conventions of Weatherburn's *Differential geometry*, if

$$\mathbf{r} = (x, y, z),$$

then

$$\mathbf{t} = (\dot{x}, \dot{y}, \dot{z})\theta'$$

and

$$\kappa\mathbf{n} = (\ddot{x}, \ddot{y}, \ddot{z})\theta'^2 + (\dot{x}, \dot{y}, \dot{z})\theta'',$$

whence, since $\mathbf{t} \times \mathbf{n} = \mathbf{b}$,

$$\kappa\mathbf{b} = (\dot{y}\ddot{z} - \dot{z}\ddot{y}, \dot{z}\ddot{x} - \dot{x}\ddot{z}, \dot{x}\ddot{y} - \dot{y}\ddot{x})\theta'^3,$$

and κ is the modulus of this last vector, θ' being determined from the fact that $\mathbf{t}$ is a unit vector.

To determine τ, we write

$$\mathbf{b} = (p, q, r)K,$$

where $K = (p^2 + q^2 + r^2)^{-1/2}$, taking p, q, r to be the simplest possible direction ratios of $\mathbf{b}$. Then

$$-\tau\mathbf{n} = (\dot{p}, \dot{q}, \dot{r})\theta' K + (p, q, r)\, dK/ds.$$

But

$$\mathbf{b} \times \mathbf{n} = -\mathbf{t}.$$

Hence

$$\tau\mathbf{t} = (q\dot{r} - \dot{q}r, r\dot{p} - \dot{r}p, p\dot{q} - \dot{p}q)\, \theta' K^2,$$

and comparing this with $\mathbf{t}$, τ is determined both in magnitude and sign. Of

course, only one component of this last vector product need be found, and the value of θ' is not required if τ only is wanted.

Example. $\mathbf{r} = a(\cos 2\theta, \sin 2\theta, 2\sin\theta)$,

so that $\mathbf{t} = 2a(-\sin 2\theta, \cos 2\theta, \cos\theta)\theta'$.

Hence

$$\kappa\mathbf{n} = 2a(-2\cos 2\theta, -2\sin 2\theta, -\sin\theta)\theta'^2 + 2a(-\sin 2\theta, \cos 2\theta, \cos\theta)\theta'' :$$

whence

$$\kappa\mathbf{b} = 4a^2(-\cos 2\theta \sin\theta + 2\sin 2\theta\cos\theta, -2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta, 2)\theta'^3$$
$$= 4a^2\theta'^3(\sin\theta + \sin 2\theta\cos\theta, -\cos\theta - \cos 2\theta\cos\theta, 2)$$

and since $\mathbf{t}$ is a unit vector,

$$2a\theta'(1+\cos^2\theta)^{1/2} = 1.$$

Thus $$\kappa = (5+3\cos^2\theta)^{1/2}/2a(1+\cos^2\theta)^{3/2}.$$

Putting the components of $\mathbf{b}$ in a form more suitable for differentiation, and writing $K = (5+3\cos^2\theta)^{-1/2}$, we have

$$\mathbf{b} = (3\sin\theta - 2\sin^3\theta, -2\cos^3\theta, 2)K$$

and

$$-\tau\mathbf{n} = (3\cos\theta - 3\sin 2\theta\sin\theta, 3\sin 2\theta\cos\theta, 0)\theta'K$$
$$+ (3\sin\theta - 2\sin^3\theta, -2\cos^3\theta, 2)\,dK/ds,$$

and the first component of the vector product is

$$-6\theta'K^2\sin 2\theta\cos\theta.$$

Comparing this with the first component of $\mathbf{t}$,

$$\tau = -6K^2\sin 2\theta\cos\theta/(-2a\sin 2\theta)$$
$$= 3\cos\theta/a(5+3\cos^2\theta).$$

D. A. T. WALLACE.

2401. *Two geometrical fallacies.*

1. In Note 2190 (February, 1951), Dr. Maxwell invoked Algebra to prove that a conic is a line-pair. It is fitting that Geometry should provide a proof of the converse proposition : *A line-pair is a (proper) conic.*

This depends on the theorem that the locus of the meet of corresponding lines of two coplanar projective pencils with different centres is a proper conic if the common line is not self-corresponding.

Let P be the pole of a fixed line u for a circle of a coaxal system with limiting points L, M. Then the polars of P for the circle and for the point-circles L, M concur at Q on u. Since LQ is perpendicular to LP and MQ to MP, we have, using the sign of equality to denote " is projective with ",

$$L(P) = L(Q) = (Q) = M(Q) = M(P),$$

which gives the construction for MP or LP when the other is known.

This construction makes LM self-corresponding only when u is perpendicular to LM. In fact, when u passes through M, when LP is LM, MP is perpendicular to u, and when MP is ML, LP is perpendicular to LM. Hence in this last case, as LM is not self-corresponding, the locus of the pole of u for the coaxal circles is a proper conic. But, as is easily seen, when u passes through M, its poles for the system lie on the two lines mentioned, the perpendicular

at L to LM and the perpendicular at M to u (containing its poles for the point-circle M).

Hence the line-pair is a proper conic.

2. To find the number of conics passing through three points A, B, C and touching two lines u, v, take B,C as absolute points (or project into the circular points at infinity). A circle through A touching u, v will also pass through the image of A in the bisector of the angle of u, v containing A. There are then *two* circles of the coaxal system through these two points touching u and therefore also v. Hence there are two conics through A, B, C touching u, v.

The fallacy in this is easily seen and corrected, and the point would not be worth mentioning but that the corresponding process gives the correct solution to the dual problem of the number of conics through two given points U, V touching three given lines a, b, c. If U, V are taken as absolute points, we get the four circles touching the sides of the triangle abc.

It is rather amusing to see how an algebraical argument could prove misleading in the last case. On the ground that the conditions that a conic should pass through a point or touch a line give respectively relations of the first and second degrees in the coefficients of its equation, one might suppose there were $2 \times 2 \times 2 = 8$ conics satisfying the conditions, especially as this reasoning gives the correct result in the case of two lines and three points.

H. V. MALLISON.

2402. *Approximating to an ellipse by circular arcs.*

Various methods for drawing a series of circular arcs to form an approximate ellipse have appeared in texts on Engineering Drawing and in the *Gazette*. The most accurate of those which the writer has seen is as follows. For a three-arc approximation to the first quadrant portion of an ellipse with centre O, semi-major axis $OA = a$, and semi-minor axis $OB = b$, draw the circles of curvature at A and B and a " transition circle " touching these two circles and with radius $\sqrt{(ab)}$, the *geometric* mean of their radii.

If b/a is small, even this construction may give rather poor results ; an obvious improvement would be to draw the *true* circle of curvature of radius $\sqrt{(ab)}$ and centre C, and *two* transition circles with appropriate mean radii, thus obtaining a five-arc approximation. Verification of the following simple constructions for locating C provides an interesting exercise for students of elementary calculus.

(i) By standard construction, locate D on OB produced, such that

$$OD = \sqrt{(ab)} ;$$

then OC is of length $a - b$ and parallel to DA. (The corresponding point P on the ellipse is such that $CP = \sqrt{ab}$ is perpendicular to OC.)

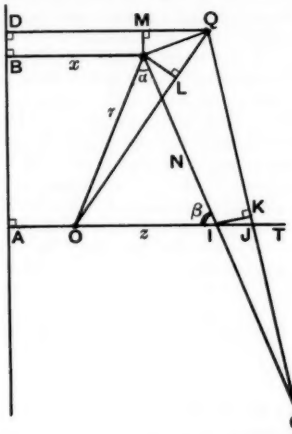
(ii) Alternatively, it can be shown that $\tan \phi = \sqrt{(b/a)} = (\sqrt{(ab)}/a$, where ϕ is the eccentric angle of P. The standard auxiliary circle construction then locates P and the tangent at P ; C is distant $\sqrt{(ab)}$ from P along the normal.

A. W. WALKER.

2403. *The centre of curvature of a conic.*

Attention has been called (*Gazette*, XXXV, 1951, p. 19 ; see also Salmon, *Conic Sections*, 6th edition, 1924, pp. 228–9) to the expression $R = N \sec^2 \alpha$ for the radius of curvature of a conic, and to the resulting elegant construction for locating the centre of curvature. Here is an alternative proof of this expression, using the basic geometrical definition of curvature. In the figure, P and Q are adjacent points on any curve, O is any fixed point, and ABD any

fixed line. The lines TOA, PB and QD are perpendicular to ABD; and PIC, QJC are the normals to the curve at P and Q. The symbols x, r, z, N denote the distances BP, OP, OI and IP respectively. Then, where R is the radius of



curvature at P and " lim " means " the limit as Q approaches P ", we have

$$\lim LQ/PQ = \sin\alpha, \qquad \lim MQ/PQ = \sin\beta,$$

so
$$\frac{z}{r} = \frac{\sin\alpha}{\sin\beta} = \lim\frac{LQ}{MQ} = \frac{dr}{dx}$$

and
$$\frac{R}{N} = \lim\frac{IP}{CP} = \lim\left(1 - \frac{CI}{CP}\right)$$

$$= \lim\left(1 - \frac{IK}{PQ}\right) = \lim\left(1 - \frac{IJ\sin\beta}{LQ/\sin\alpha}\right)$$

$$= 1 - \sin\alpha\sin\beta\,.\,\frac{dz}{dr} = 1 - \sin^2\alpha\,.\,\frac{dx}{dr}\cdot\frac{d}{dr}\left(r\frac{dr}{dx}\right),$$

a quite general result. Now, if the curve is a conic with O as focus and ABD as corresponding directrix, then $r = ex$ and $N = R\cos^2\alpha$. Various expressions for R, and corresponding constructions, can be deduced by introducing the focal radii: the only one which compares favourably with the above is $R = 2r\sec\alpha$ for the parabola.

An entirely different construction (Salmon, p. 229) is of interest because it does not depend on an expression for R and follows at once from purely geometrical reasoning. If $b \neq a$, the equation

$$ax^2 + by^2 + gx + fy + c + k(y - mx - p)(y - m'x - p') = 0$$

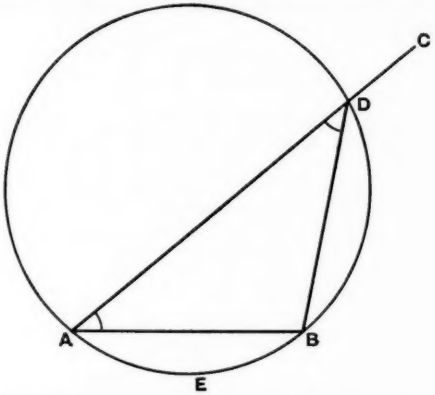
can represent a circle only if $m' = -m$; *i.e.* if P, P_1, P_2, P_3 are four concyclic points on a conic, then the straight lines PP_1, P_2P_3 are equally inclined to the focal axis. Considering the limiting case when P_2 and P_3 coincide with P, we deduce that the common chord and tangent of the conic and its circle of curvature at P are equally inclined to the focal axis. The line joining the centre of the conic to the image point of P in the focal axis cuts this common

chord at its midpoint M ; the perpendicular to the chord at M cuts the normal at the required centre of curvature.

Much additional information on this subject will be found in Carr, *Synopsis of Elementary Results in Pure Mathematics*, (1886), pp. 252–255, and in the *Encyk. der Math. Wiss.*, III, 2 (1), pp. 69–79. A. W. WALKER.

2404. *A circle construction.*

The following method for constructing a segment of a circle on a given line to contain a given angle was devised by I. H. Munro (aged 13) of William Hulme's Grammar School, Manchester. I have not seen it in any textbook.



Given the line AB, construct $\angle CAB$, equal to the given angle, mark D on AC such that $BD = AB$, and draw the circumcircle of the triangle ABD. The segment ABD is the required segment. If the given angle is obtuse, say the supplement of $\angle BAC$, a similar construction gives AEB as the required segment. D. G. R. MARTIN.

2405. *Notes on conics.* **17.** *Salmon's Theorem.*

If AP is the perpendicular from A on the polar of B, for a circle whose centre is O, and BQ the perpendicular from B on the polar of A, then $OA \cdot BQ = OB \cdot AP$.

There is no ambiguity of sign in this theorem, since the products on each side are of parallel lengths, but the proof from the equation of the polar needs more attention to this detail than is commonly given. The purpose of this note is to express the result differently. Let the line AB cut the polar of A in F and the polar of B in G. Then $AF/BF = AC/BQ$, $AG/BG = AP/BD$, where AC, BD are the perpendiculars from A, B on their own polars. Hence Salmon's theorem is equivalent to

$$\frac{AF \cdot AG}{BF \cdot BG} = \frac{AO \cdot AC}{BO \cdot BD};$$

that is to say,

If the line AB cuts the polars of A and B in F and G, then the ratio of the powers of A and B for the circle is

$$\frac{AF \cdot AG}{BF \cdot BG}.$$

The theorem in this form has an important application in the metrical geometry of the conic. If a line l through a point O cuts a conic in P and Q consideration of the individual lengths OP, OQ naturally gives way to consideration of their sum and product, and if we are to study these magnitudes freely, we want to replace them by magnitudes derived from O and l and existing whether or not the line does cut the conic. For the sum, the solution is easy. If V is the midpoint of PQ, then $OP+OQ=2OV$, and V can be identified as the point in which l cuts the diameter to which it is an ordinate; this point V always exists, and while theorems in which OV is involved can be enunciated as general theorems, they will be theorems about the arithmetic mean of OP and OQ whenever the points P, Q exist.

For the product, there are two solutions. If l cuts the directrix in R and SR cuts the eccentric circle of O in p, q, then

$$\frac{OP \,.\, OQ}{OR^2}=\frac{Sp \,.\, Sq}{Rp \,.\, Rq}=\frac{\pi_S}{\pi_R},$$

where π_S, π_R are the powers of S, R for this circle. Alternatively, if l cuts the polar of O for the conic in U, then OU is the harmonic mean of OP and OQ, and $OP \,.\, OQ=OU \,.\, OV$. Whether or not l cuts the conic, the two magnitudes $OU \,.\, OV$ and $(\pi_S/\pi_R)OR^2$ both exist, and when l cuts the conic these two magnitudes have been proved equal, since each is equal to the product $OP \,.\, OQ$, but if we are to use one or the other to replace the product, we must satisfy ourselves that the two always are equal. This is the step we can now take. If the lines through O parallel to SU, SV cut SR in u, v, then immediately from the parallelisms,

$$\frac{OU \,.\, OV}{OR^2}=\frac{Su \,.\, Sv}{Ru \,.\, Rv},$$

and since u is on the polar of R for the eccentric circle and v is on the polar of S, this ratio is π_R/π_S. If θ is an angle between l and the focal axis, the radius of the eccentric circle is $e_\theta \mid OR \mid$, where e_θ is the oblique eccentricity $e \mid \cos\theta \mid$. Hence $\pi_R=OR^2(1-e_\theta{}^2)$, and the power of O along l can be defined either as $OU \,.\, OV$ or as $\pi_S/(1-e_\theta{}^2)$. The definition as $OU \,.\, OV$ fails if O is itself the midpoint V, for then the polar of O is parallel to l and the factor OU becomes infinite as OV becomes zero. If O is on the conic, OU is zero but OV is not necessarily infinite, and the power is as a rule definitely zero; in the alternative definition, π_S is zero if the eccentric circle passes through S, and the power along l is then zero or indeterminate according as the denominator $1-e_\theta{}^2$ is not or is zero, that is, according as l is not or is parallel to an asymptote.

Carnot's theorem combines with the form of Salmon's theorem we are using to give the theorem that

If two triangles are conjugate for a conic, the six intersections of non-corresponding sides lie on a conic, and the six joins of non-corresponding vertices touch a conic.

The use of Salmon's theorem implies that the result is established first for the case in which the conic with respect to which the triangles are conjugate is a circle, but the theorem is projective and can be generalised immediately.

E. H. N.

2406. *A science lesson.*

OBJECT. To find the speed of raindrops.

[It is assumed that the rain is falling vertically. Corrections for a known angle of fall may easily be applied.]

APPARATUS. 1 sheet of paper, gummed on one side;
1 pencil, with fine point;
1 ruler;
1 railway engine, with carriage.

METHOD. Stick paper on inside of (closed) carriage window. Draw horizontal line AB, of length 6 in., near foot of paper, and draw vertical line BY through B.

Take train, and impart speed of 30 m.p.h. along a level track.

With ruler (and pencil) draw the line through A parallel to direction of apparent fall of rain. Let this line meet BY in C. Measure BC, in inches.

CALCULATION. The speed with which the rain is falling is $5BC$, in miles per hour.

PRINCIPLE ILLUSTRATED. The triangle of velocities.

E. A. MAXWELL.

2407. *A note on equivalent product theorems.*

In a recent paper,* W. N. Bailey gave the result

$$\prod_{n=1}^{\infty}(1+q^n/z)(1+q^{n-1}z)(1-q^{2n-1}/z^2)(1-q^{2n-1}z^2)(1-q^n)$$
$$=\prod_{n=1}^{\infty}(1-q^{3n-2}z^3)(1-q^{3n-1}/z^3)(1-q^{3n})$$
$$+z\prod_{n=1}^{\infty}(1-q^{3n-2}/z^3)(1-q^{3n-1}z^3)(1-q^{3n}) \quad \ldots\ldots(1)$$

He told me that it could not apparently be deduced from the more general result†

$$\Pi(b/a, aq/b, df/a, aq/df, ef/a, aq/ef, bde/a, aq/bde)$$
$$=\Pi(f/a, aq/f, bd/a, aq/bd, be/a, aq/be, def/a, aq/def)$$
$$-(b/a)\Pi(d, q/d, e, q/e, f/b, qb/f, bdef/a^2, a^2q/bdef) \quad \ldots\ldots(2)$$

where

$$\Pi(a, b, c, \ldots, h) \equiv \prod_{n=0}^{\infty}(1-aq^n)(1-bq^n)\ldots(1-hq^n).$$

It will now be shown, however, that (1) can be deduced from (2). Put q^2 for q, and then write q for a, qz for b, z for d, qz for e, and q^2 for f in (2). Then

$$\prod_{n=1}^{\infty}(1-q^{3n-3}z)(1-q^{3n}/z)(1-q^{3n-2}z)(1-q^{3n-1}/z)(1-q^{3n-1}z)(1-q^{3n-2}/z)$$
$$\times(1-q^{3n-2}z^3)(1-q^{3n-1}/z^3)$$
$$-\prod_{n=1}^{\infty}(1-q^{3n-2})(1-q^{3n-1})(1-q^{3n-3}z^2)(1-q^{3n}/z^2)(1-q^{3n-2}z^2)$$
$$\times(1-q^{3n-1}/z^2)(1-q^{3n-1}z^2)(1-q^{3n-2}/z^2)$$
$$=z\prod_{n=1}^{\infty}(1-q^{3n-3}z)(1-q^{3n}/z)(1-q^{3n-2}z)(1-q^{3n-1}/z)(1-q^{3n-2}/z)$$
$$\times(1-q^{3n-1}z)(1-q^{3n-1}z^3)(1-q^{3n-2}/z^3).$$

On combining the various groups of products together and multiplying across by

$$\prod_{n=1}^{\infty}(1-q^{3n})/\{(1-q^n/z)(1-q^{n-1}z)\}$$

* Bailey (1). † Bailey, (2), (5.2).

we obtain (1).

By proceeding in a manner similar to that used by Bailey to deduce (2), but using ${}_{10}\Phi_9$ series instead of ${}_8\Phi_7$ series, I have deduced the more general formula

$$ab\,\Pi(bc/a,\ aq/bc,\ bd/a,\ aq/bd,\ be/a,\ aq/be,\ bf/a,\ aq/bf,\ g/a,\ aq/g,\ h/a,\ aq/h,\ g/h,\ hq/g)$$
$$-ab\,\Pi(ch/a,\ aq/ch,\ dh/a,\ aq/dh,\ eh/a,\ aq/eh,\ fh/a,\ aq/fh,\ b/a,\ aq/b,\ g/a,\ aq/g,\ g/b,\ bq/g)$$
$$-ag\,\Pi(cg/a,\ aq/cg,\ dg/a,\ aq/dg,\ eg/a,\ aq/eg,\ fg/a,\ aq/fg,\ b/a,\ aq/b,\ h/a,\ aq/h,\ b/h,\ hq/b)$$
$$+bh\,\Pi(c,\ q/c,\ d,\ q/d,\ e,\ q/e,\ f,\ q/f,\ b/h,\ hq/b,\ g/h,\ hq/g,\ g/b,\ bq/g)$$
$$=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

provided that $a^3q^3 = bcdefgh$.

This is equivalent to a relation between four products each containing seven theta-functions, and involves six independent parameters, for if we replace b, g, h by ab, ag, ah, the parameter a disappears from the formula. The special case in which $c = a^3q/bdef$, $g = f$ and $h = aq/f$ gives (2).

It has been pointed out to me by Professor Bailey that (2) and (3) are only disguised forms, when $n = 3, 4$, of the general theorem* on sigma functions,

$$\sum_{r=1}^{n} \frac{\sigma(a_r - b_1)\,\sigma(a_r - b_2)\ldots\ldots\sigma(a_r - b_n)}{\sigma(a_r - a_1)\,\sigma(a_r - a_2)\ldots\ldots\sigma(a_r - a_{r-1})\,\sigma(a_r - a_{r+1})\ldots\ldots\sigma(a_r - a_n)} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where $a_1 + a_2 + a_3 + \ldots + a_n = b_1 + b_2 + b_3 + \ldots + b_n$. Since

$$\sigma(z) \equiv 2\omega_1 \exp\left[(\eta_1 z^2 + \pi i z)/2\omega_1\right] \times \prod_{n=1}^{\infty} \frac{(1 - \exp(\pi i z/2\omega_1)q^{2n})(1 - \exp(-\pi i z/2\omega_1)q^{2n-2})}{(1-q^{2n})^2}$$

in the usual notation, putting a_r for $\exp(\pi i a_r/2\omega_1)$, b_r for $\exp(\pi i b_r/2\omega_1)$, $q^{\frac{1}{2}}$ for q, we have

$$\sum_{r=1}^{n} \frac{\Pi(a_r/b_1,\ a_r/b_2,\ \ldots\ldots\ a_r/b_n, qb_1/a_r, qb_2/a_r,\ \ldots\ldots\ qb_n/a_r)}{\Pi(a_r/a_1,\ a_r/a_2,\ \ldots\ldots\ a_r/a_{r-1},\ a_r/a_{r+1},\ \ldots\ldots\ a_r/a_n,\ qa_1/a_r,\ qa_2/a_r,\ \ldots\ldots,\ qa_{r-1}/a_r,\ qa_{r+1}/a_r,\ \ldots,\ qa_n/a_r)} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where $a_1a_2a_3 \ldots\ldots a_n = b_1b_2b_3 \ldots\ldots b_n$.

This relation involves n products each containing $\frac{1}{2}n(n-1)+1$ theta-functions, and there are $2n-2$ independent parameters. If $n = 4$, $a_1 = b/a$, $a_2 = h/aq$, $a_3 = g/aq$, $a_4 = 1$, $b_1 = 1/c$, $b_2 = 1/d$, $b_3 = 1/e$ and $b_4 = 1/f$, we again have (3) above.

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L. J. SLATER.

* Whittaker, (3), p. 451, ex. 3.

2408. *Note on matrices.*

In Turnbull and Aitken's *Canonical Matrices* (1932, p. 97), we read : If A is a matrix of order $m \times n$, with non-zero columns of complex elements, the Hermitian matrix $\bar{A}'A$ is positive definite if $m > n$, and non-negative definite if $m < n$.

The statement is not quite correct, and the line of proof suggested (*loc. cit.*, ex. 1) seems unnecessarily elaborate. In fact, if $Ax = y$, $\bar{x}'\bar{A}'Ax = \bar{y}'y \geqq 0$. Hence $\bar{A}'A$ is always non-negative. Since its order is $n \times n$, it will be positive-definite if, and only if, its rank is n. Now its rank is the same as that of A. This follows from the fact that the vectors annihilated by the two matrices respectively are identical. In fact, if $Ax = 0$, $\bar{A}'Ax = 0$; and conversely, if $\bar{A}'Ax = 0$, $0 = \bar{x}'\bar{A}'Ax = \bar{y}'y$, where $y = Ax$, hence $Ax = 0$. The only effect of zero columns in A is to reduce its rank. Hence the correct statement would seem to be : " Non-negative definite if the rank of A is less than n, (which always happens when $m < n$, but may happen when $m \geqq n$), and positive-definite when the rank is n." M. F. EGAN.

2409. *Bridge problem.*

Rouse Ball (*Mathematical Recreations and Essays*, eleventh edition, 1939, p. 297) discusses briefly the problem of arranging $4m$ members of a bridge club for $4m - 1$ rounds of m rubbers so that (i) no two members shall play as partners more than once, (ii) each member shall meet each other member as opponent twice (once on his left and once on his right would be a desirable refinement, but is probably impossible). Only empirical solutions are quoted, with incomplete references. Kraïtchik (*La Mathématique des Jeux*) mentions the extension to $4m + 1$ players, one standing down in each round, and gives an imperfect solution for 17 players, satisfying only condition (i).

In case $4m + 1$ has all its prime factors $\equiv 1 \pmod 4$, so that the congruence

$$x^2 \equiv -1 \pmod{4m+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

is soluble, the problem of $4m + 1$ players is very simply soluble. Denote the players by the residues $0, 1, \ldots, 4m \pmod{4m+1}$; then a typical rubber is

$$(u + v,\ xu + v,\ -u + v,\ -xu + v), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

the players being arranged in the order of play. v is constant in each round, and by running through a complete set of residues it gives the $4m + 1$ rounds. u may be considered as the number of the seat, and by running through a set of residues excluding zero it gives m rubbers (the odd man out being v). Each of these comes four times over, since the arrangement (2) is only permuted cyclically if u is replaced by xu, $x^2u \equiv -u$, or $x^3u \equiv -xu$. To find the seat and the round in which player y will have z as his partner, we solve, uniquely,

$$u + v \equiv y, \qquad -u + v \equiv z \pmod{4m+1}.$$

Similarly, y will have z as his left-hand opponent when he (y) sits in seat u, in round v, given by

$$u + v \equiv y, \qquad xu + v \equiv z.$$

For the case of $4m$ players, when m is prime to six, the following arrangement serves ; the players are denoted, to avoid complicated suffixes, by $A(x)$, $B(x)$, $C(x)$, $D(x)$, with numbers x reduced (mod m) :

Part I (m *rounds*).

(i) $A(u)$, $C(u)$, $B(u)$, $D(u)$.
(ii) $A(u+v)$, $B(u+2v)$, $A(u-v)$, $B(u-2v)$.
(iii) $C(u+v)$, $D(u+2v)$, $C(u-v)$, $D(u-2v)$.

Part II (*m rounds*).
(i) $A(u)$, $D(u)$, $C(u)$, $B(u)$.
(ii) $A(u+v)$, $A(u-v)$, $C(u-2v)$, $C(u+2v)$.
(iii) $B(u-v)$, $B(u+v)$, $D(u+2v)$, $D(u-2v)$.

Part III (*m rounds*).
(i) $A(u)$, $B(u)$, $D(u)$, $C(u)$.
(ii) $A(u-v)$, $A(u+v)$, $D(u-2v)$, $D(u+2v)$.
(iii) $B(u+v)$, $B(u-v)$, $C(u+2v)$, $C(u-2v)$.

Part IV ($(m-1)/2$ *rounds*).
$A(u+v)$, $C(u+2v)$, $B(u-v)$, $D(u-2v)$.

Part V ($(m-1)/2$ *rounds*).
$A(u-v)$, $D(u+2v)$, $B(u+v)$, $C(u-2v)$.

Here u runs through a complete set of residues (mod m) ; in parts I to III its variation gives the m rounds, and it is the same for all the tables in any one round. v runs through the residues $1, 2, \ldots, (m-1)/2$ only ; in sections (ii) and (iii) of parts I to III, the seats may be identified by the letter A, B, C, D and the value (and coefficient) of v. In parts IV and V, u varies from table to table, and v from round to round. That this arrangement satisfies the conditions may be seen as above. Section (i) of parts I to III obviously provides for the partnerships and oppositions of players with the same number. For the rest, note that, for example, $A(y)$ and $A(z)$ will be partners in section (ii) of part I, in round u and at tables that may be described as A_v and A_{-v}, where

$$u \pm v \equiv y, \qquad u \mp v \equiv z \pmod{m}.$$

With the restriction on v, these congruences can be solved for just one choice of the ambiguous signs.

To derive an arrangement for $4m+1$ players, again with m prime to 6, we add one more player E to the above arrangement and modify it as follows :

(*a*) Delete I (i), II (i), and III (i).

(*b*) Rewrite IV and V as IV (ii) and (iii) by treating u and v as in I to III ; that is, let u vary for the rounds and v for the tables.

This leaves $A(u)$, $B(u)$, $C(u)$ and $D(u)$ unaccounted for.

(*c*) Complete as follows :

I (i) E, $B(u)$, $A(u)$, $D(u)$.
II (i) E, $D(u)$, $B(u)$, $C(u)$.
III (i) E, $A(u)$, $C(u)$, $B(u)$.
IV (i) E, $C(u)$, $D(u)$, $A(u)$.
V (*one round, u identifying the table*).

$$A(u),\ B(u),\ D(u),\ C(u).$$

(This part of the arrangement is based, as may easily be verified, on an arrangement for five players A, B, C, D, E.)

It would make this note too long to complete the solution for an even number of players. But this can be done by first showing that a solution for $16m$ or $36m$ players can be derived from one for $4m$, and then writing down schemes for $8m$, $12m$ and $24m$ players, again with m prime to six.

I could not obtain a complete solution for an odd number of players ; but the case of $2^r \cdot 3^s \cdot m+1$, $(6, m)=1$, may be reduced to that of 2^r+1 or $2^r \times 3+1$ according as s is even or odd. Difficulty arises only if $4m+1$ is congruent to 1 to modulus a high power of 2 ; but can be avoided if it has a factor that is not of such a form.

G. L. WATSON.

2410. *A simple method for the approximate estimation of standard deviations.*

The apparatus required consists of two pins with large heads and a modified ten inch set-square, whose shorter sides are divided into any convenient scales with origin at the right-angle. For universal use, an opaque celluloid is suggested, one face divided in inches and tenths, and the other in cms. and mms.

An origin is taken towards the lower left of a sheet of paper and one of the pins inserted therein. Using an appropriate scale, such that the largest deviation is represented by 4–5 inches, the point corresponding to this deviation is placed against the origin pin with the left hand, the hypotenuse of the square sloping upwards to the right, and the second pin held with the right hand opposite the division on the adjacent side of the square, to which the next largest division corresponds (to the same scale and paying no regard to sign).

With the left hand the square is now manipulated so that the 90° corner is against the right-hand pin while the side originally in contact with the origin pin is kept in contact with it. The right-hand pin is then advanced along the second side to the position corresponding to the third deviation. This process is continued for the remaining deviations.

When this is completed, the distance from the origin to the final position of the right-hand pin is measured and, to the same scale as the deviations, gives the square root of the sum of the squares of the deviations. If this is divided by the square root of the number of observations less one (or 0 or 2 as appropriate in some cases) one movement of a slide-rule performs the divisions and gives both the standard deviation and the variance at one operation, while, to the accuracy to be expected from the method, the probable error may be taken as two-thirds of the standard deviation.

The proof of the method is obvious, being merely a repeated application of Pythagoras' theorem.

The accuracy of the method is limited, apart from errors in manipulation, by the fact that the sides of the set-square are tangents to the diameters of the pins, whereas for theoretical purposes they are assumed to pass through their centres; however, with fine pins and, say, a ten inch set-square, this error, even though cumulative, is probably negligible.

By the use of scales in both inches and cms., it should be possible to find a scale related to the deviations by a factor of 10 only, which eliminates mistakes due to doubling or halving.

This method has been tried by a colleague, who normally does statistical work with an electric desk calculator; on his first trial, he reports that, for 2-figure accuracy, the method is at least as rapid as the use of the calculating machine.

D. L. GOLDBY.

2411. *The integration of the school curriculum in mathematics.*

The suggestion has often been made that the school curriculum in mathematics should be unified and integrated; that it should not be divided into water-tight compartments, Arithmetic, Algebra and Geometry; that the concept of "general" mathematics, analogous to general science, is a fruitful one. The Norwood Report gives support to this view when it says,[1] "This ... course from 11 plus to 16 plus should include, we believe, parts of Arithmetic, Algebra, Graphs, Geometry and Trigonometry, treated not as isolated subjects but with the fullest measure of coordination". On the same question, the Scottish Report on Secondary Education says,[2] "Another symptom of our excessive preoccupation with the ordering of mathematical truth as against the application of it to the real world is the almost invariable practice of teachers and textbooks of dividing Mathematics into branches or subjects. This may be congenial to the mathematical thinker but its pedagogic results

are bad. It breaks up the concrete unity of the real, obscures relationships by treating connected ideas in isolation, overloads certain branches of the subject and delays too long the introduction of others that are of singular interest and practical importance ". It goes on to say with great emphasis,[3] " We deplore the practice of treating Mathematics as so many separate branches ". The Interim Report of the Council for Curriculum Reform makes the following observation on the same question,[4] " . . . it is suggested that the division of mathematics into geometry, algebra, arithmetic, etc. should be broken down and symbolism treated as a powerful means for the discussion and solution of real problems ". Lastly, we may quote the words of the Jeffery Report,[5] " . . . the traditional divisions of mathematics . . . were harmful and restrictive in teaching practice ", and again,[6] " In relation to Elementary Mathematics an overwhelming case can be made out for the fusion of mathematical subjects and in particular for the close association of geometry and trigonometry ". Not one of these Reports, however, develops any argument or cites evidence in support of the suggestion that Arithmetic, Algebra, and Geometry can be fused into one. They ignore the existence of a considerable body of psychological opinion pointing in the opposite direction. A more positive accusation that can be made against them is that not one of them goes so far as to suggest any principle on the basis of which these subjects may be unified or integrated.

As a sequel to the Reports, textbooks have appeared with titles that give the impression that they have broken down the old barriers and deal with mathematics as a unity. Their contents generally disappoint. The subject matter that was in the past offered in separate books on Arithmetic, Algebra, and Geometry is now offered in separate chapters but in the *same* book, and in this lies the alleged unification of the mathematical curriculum.

Two questions arise in connection with the problem of unifying the school curriculum in mathematics. They are: 1. Do Arithmetic, Algebra, and Geometry call for distinct unconnected abilities or do these abilities have anything in common? 2. If they have anything in common, how can this common core be utilised to integrate these subjects for teaching purposes?

Research findings on the first of these questions are somewhat inconclusive. On one side is the weighty opinion of the late Professor Charles Spearman who, using statistical techniques to analyse the results of experimental investigations, wrote as follows:[7] " There appears, then, no real basis for the common opinion which would take arithmetic and geometry to furnish one single ability. Their union as ' mathematics ' seems, rather, to be merely one of practical convenience "; and more recently, of H. W. Oldham,[8] who, reporting a research spread over a period of five years, expressed the view that Arithmetic, Algebra, and Geometry seemed to have no large enough group factor to justify their being placed in one class for purposes of examination. On the other side are William Brown, who, basing his conclusions on a quantitative analysis of mathematical ability, said,[9] " . . . the balance of evidence seems to be in favour of the existence of a special capacity or faculty, underlying mathematical ability, distinct from, and with no essentially close connection with other forms of intellectual capacity ", and J. H. Wilson,[10] who examined the data taken from the performances of 371 children at the School Certificate Examination of the Joint Board of the Northern Universities and found that the correlations between Arithmetic, Algebra, and Geometry were so high that they could be explained only by postulating a group factor operating in mathematics. The characteristics of the special capacity that Brown speaks of and the group factor that Wilson speaks of are not revealed by their quantitative analyses or in their discussions. An answer to the second question formulated above must therefore be sought in the writings of mathematical philosophers.

The concept of functionality, first adumbrated by Felix Klein who says,[11] "It is my conviction that the function concept . . . should be the soul of mathematical study in the schools," and implicit in most of T. P. Nunn's writings,[12] but brought to the surface, its logical and psychological foundations examined, and used in a teaching programme by H. R. Hamley,[13] suggested itself to the writer as providing a fruitful line of investigation. It was thought that if tests in Arithmetic, Algebra, and Geometry based on the function concept were given to a group of children and the results analysed by modern factorial techniques some light would be thrown on the existence (or, non-existence) of factors common to these three subjects. The results of the study[14] show that when attention is concentrated on the functional aspects of Arithmetic, Algebra, and Geometry they do not stand apart in solitary isolation but group together according to the fundamental concepts involved. There appears, therefore, to be a prima facie case for regarding the concept of functionality as a principle suggestive for the integration of the mathematical curriculum. The possibilities of the function concept deserve careful exploration by the teacher of mathematics and the framer of curricula, and Hamley's pioneer work merits their attention. E. T. Norris has, in a recent article,[15] described the way in which the concept of functionality was used as the basis of a Training College Mathematics course. It is possible that a mathematical curriculum unified round the function concept would perhaps provide a complete answer to Spearman's observation that the union of Arithmetic and Geometry as mathematics is merely one of practical convenience.

J. E. JAYASURIYA.

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2412. *A new proof of the multinomial theorem.*

If n is a positive whole number, $(x_1+x_2+\ldots+x_m)^n$ is a homogeneous polynomial

$$\sum c(p_1, p_2, \ldots, p_m)x_1^{p_1}x_2^{p_2}\ldots x_m^{p_m}.$$

The problem is, given m numbers $p_1, p_2, \ldots, p_m$ whose sum is n, to find the coefficient $c(p_1, p_2, \ldots, p_m)$.

If $n<m$, there is no term of the polynomial which includes all the variables. The coefficient of $x_1x_2\ldots x_m$ in the expansion of $(x_1+x_2+\ldots+x_m)^m$ is a number $\phi(m)$ dependent only on m. Since there is no term of $(x_1+x_2+\ldots+x_m)^{m-1}$ which includes all the variables, the term of $(x_1+x_2+\ldots+x_m)^m$ which includes all the variables arises from the sum

$$x_1(x_2+x_3+\ldots+x_m)^{m-1}+x_2(x_1+x_3+\ldots+x_m)^{m-1}+\ldots.$$

Hence $\phi(m)=m\phi(m-1)$, and since $\phi(1)=1$, it follows that $\phi(n)=n!$.

In the function $(x_1+x_2+\ldots+x_m)^n$, replace each variable x_k by the sum of p_k variables $x_1^{(k)}, x_2^{(k)}, \ldots, x_{p_m}^{(k)}$, where $p_1, p_2, \ldots, p_m$ is the set of indices for which the coefficient is to be determined. In any term

$$c(r_1, r_2, \ldots, r_m)x_1^{r_1}x_2^{r_2}\ldots x_m^{r_m}$$

of the expansion for $(x_1+x_2+\ldots+x_m)^n$ except the term

$$c(p_1, p_2, \ldots, p_m)x_1^{p_1}x_2^{p_2}\ldots x_m^{p_m},$$

one or more of the indices $r_1, r_2, \ldots, r_m$ is smaller than the corresponding index in the set $p_1, p_2, \ldots, p_m$, since the sums $r_1+r_2+\ldots+r_m$, $p_1+p_2+\ldots+p_m$ are equal. Hence the term of $(x_1^{(1)}+x_2^{(1)}+\ldots+x_{p_m}^{(m)})^n$ which includes all the variables $x_1^{(1)}, x_2^{(1)}, \ldots, x_{p_m}^{(m)}$ comes entirely from the one term

$$c(p_1, p_2, \ldots, p_m)x_1^{p_1}x_2^{p_2}\ldots x_m^{p_m},$$

and therefore

$$\phi(n)=c(p_1, p_2, \ldots, p_m)\phi(p_1)\phi(p_2)\ldots\phi(p_m).$$

That is,

$$c(p_1, p_2, \ldots, p_m)=n!/p_1!\,p_2!\ldots p_m!.$$

E. H. NEVILLE.

2413. *On Note* 2186 *: Sylvester's law of nullity.*

In the Note mentioned, Mr. Dirac set out to give a short proof of the law of nullity stated in the following form : If $AB=C$ where A, B, and C are matrices of dimensions $m\times n$, $n\times p$, $m\times p$ and ranks r, s, t respectively, then

$$t\leqq r, \quad t\leqq s \qquad \ldots\ldots(1)$$

and

$$t\geqq r+s-n. \qquad \ldots\ldots(2)$$

His proof of the inequalities in (1) is correct (although r and s should be interchanged in the presentation). However, the proof of (2) is invalid since he

deduces (2) from the inequality $n-r \geqq p-t$ which is false as the following trivial example shows :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

A valid argument using Mr. Dirac's point of view can be formulated, however. The first step is the observation that a solution x of $Cx=0$ determines a solution Bx of $Ay=0$. Next, one can prove that the correspondence $x \to Bx$ maps S_{p-t}, the linear space of solutions of $Cx=0$, on to a space S_{s-t} (of dimension $s-t$) or, in other words, that as x ranges over S_{p-t} then Bx ranges over a space of dimension $s-t$. Since S_{s-t} is a subspace of S_{n-r}, the solution space of $Ax=0$, it follows that $n-r \geqq s-t$ or, $t \geqq r+s-n$ as required.

Another proof of (2) that may interest the reader is presented next. First suppose that A has the form

$$A_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r is the $r \times r$ identity matrix. Then

$$A_0 B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \quad \text{if} \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

where B_1 is the matrix consisting of the first r rows of B. Then (2) follows for the case at hand with the aid of the well-known result that if S and T are subspaces of a linear space, then $d[S]+d[T]=d[S+T]+d[S \cap T]$. Indeed, if S and T are the row spaces of B_1 and B_2 respectively, then

$$t = d[S] = d[S+T] + d[S \cap T] - d[T] \geqq s - (n-r),$$

since $d[S \cap T] \geqq 0$ and $d[T] \leqq n-r$.

For the general case, we recall that, with A of rank r, there exist non-singular matrices P and Q such that $PAQ = A_0$. Then the rank of AB is that of $(PAQ)Q^{-1}B = A_0 B_0$, let us say, where A_0 and B_0 have rank r and s respectively. This reduces the general case to the above special case and completes the proof.

ROBERT R. STOLL.

2414. *On Note* 2222.

The idea that there may be something wrong with the standard method of solution of this dynamical problem is surely ill-founded. It appears to arise from the belief that a light string can do no work. If it can do no work, then it cannot exert an impulse, for, if we suppose a mass brought from rest to velocity u by a force P, then

$$m\, dv/dt = P,$$

implying

$$m\, d(\tfrac{1}{2}v^2)/dx = P,$$

whence

$$mu = \int_0^T P\, dt,$$

$$\tfrac{1}{2}mu^2 = \int_0^X P\, dx.$$

Thus, if $\int_0^T P\, dt$ is finite, say I, so also must be $\int_0^X P\, dx$, namely $\frac{1}{2}Iu$. If we now suppose T to tend to zero, the result still remains true.

It is incorrect to say that the instantaneous stiffening of the string causes

an infinite impulse ; it causes an infinite force which produces a finite impulse and finite work.

The fact is that in mechanics problems an inextensible string must always be thought of as the limit of an extensible string, producing the same effects in an infinitely short time. H. V. LOWRY.

2415. *On Note* 2263 *: maxima and minima.*

Several correspondents have recalled that a simple example of the same type as that given by Dr. Maxwell in Note 2263 occurs in finding the maximum and minimum values of the focal radius of an ellipse. With the usual notation for this radius r, we have

$$r = a(1 - e\cos\phi) = a - ex.$$

In the first form, $dr/d\phi = 0$ at $\phi = 0$ and $\phi = \pi$, giving the minimum and maximum values of r, but from the second form $dr/dx \neq 0$ for any x. The fact is that if in a relation $r = f(u)$, u is restricted to lie in an interval (a, b) in which $dr/du \neq 0$, then $f(a)$ and $f(b)$ are the minimum and maximum values (if $dr/du > 0$). The difficulty is often encountered in statics when equilibrium positions are to be determined from maximum and minimum values of the potential energy, for here physical conditions may restrict the range of the independent variable.

2416. *The reduction of the conic and quadric to standard form.*

Consider the conic and the quadric given respectively by

$$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and

$$V = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d = 0.$$

Let Δ be the discriminant of the *quadric* and denote the cofactors in it of $a, b, c, \ldots, q, r$ by $A, B, C, \ldots, Q, R$: further denote the cofactors in D of $a, b, c, \ldots, g, h$ by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots, \mathcal{G}, H$.

It is well known that $S = 0$ represents a parabola when $\mathcal{C} = 0$ and its reduced equation is

$$(a+b)y^2 = 2\sqrt{\{-D/(a+b)\}}\,.\,x.$$

Also $V = 0$ represents a paraboloid when $D = 0$ and its reduced equation is

$$\lambda_1 x^2 + \lambda_2 y^2 = 2\sqrt{\{-\Delta/(\mathcal{A}+\mathcal{B}+\mathcal{C})\}}\,.\,z,$$

where λ_1 and λ_2 are the two non-zero roots of the discriminating cubic

$$\lambda^3 - (a+b+c)\lambda^2 + (\mathcal{A}+\mathcal{B}+\mathcal{C})\lambda = 0.$$

The textbooks neglect to prove that the expressions under the square root signs are positive. To obtain the first result, add the two identities

$$\mathcal{B}\mathcal{C} - \mathcal{F}^2 = aD, \qquad \mathcal{C}\mathcal{A} - \mathcal{G}^2 = bD.$$

Since $\mathcal{C} = 0$, the result is

$$(a+b)D = -(\mathcal{F}^2 + \mathcal{G}^2),$$

from which it follows that $-D/(a+b)$ is positive. For the second result, we refer to Jacobi's theorem on complementary minors (Ferrar, *Algebra*, p. 57) and we obtain the three equations

$$AD - P^2 = \mathcal{A}\Delta, \qquad BD - Q^2 = \mathcal{B}\Delta, \qquad CD - R^2 = \mathcal{C}\Delta.$$

Since $D = 0$, we have, by addition, that

$$(\mathcal{A}+\mathcal{B}+\mathcal{C})\Delta = -(P^2 + Q^2 + R^2).$$

Thus $-\Delta/(\mathcal{A}+\mathcal{B}+\mathcal{C})$ is positive as required. The geometrical interpretation is that the equations $S=0$ and $V=0$ with real coefficients can never represent a virtual parabola or a virtual paraboloid respectively.

Since this note was written, my colleague, Mr. V. W. Graham, has made an exhaustive search. He has found the paraboloid problem treated in Professor A. Dresden's *Solid analytical geometry and determinants*, but not by the direct method of this note. B. SPAIN.

2417. *Foci of a conic.*

With the notation of the previous note, the foci of the conic

$$S=ax^2+2hxy+by^2+2gx+2fy+c=0$$

are determined in line coordinates as the degenerate point-conics of the range

$$\phi(l,m,n)+k(l^2+m^2)\equiv\mathcal{A}l^2+\mathcal{B}m^2+\mathcal{C}n^2+2\mathcal{F}mn+2\mathcal{G}nl+2\mathcal{H}lm+k(l^2+m^2)=0\,.$$

The degeneracy condition yields

$$\mathcal{C}k^2+(a+b)Dk+D^2=0.$$

The discriminant of this quadratic equation is

$$(a+b)^2D^2-4\mathcal{C}D^2=D^2\{(a-b)^2+4h^2\}\geqslant 0,$$

which shows that the roots k_1 and k_2 are real; further, we have

$$k_1+k_2=-(a+b)D/\mathcal{C}\,,\qquad k_1k_2=D^2/\mathcal{C}\,.$$

In Sommerville's *Analytical conics* (1st ed.), p. 141, it is stated without proof that $\phi+k_1(l^2+m^2)=0$ and $\phi+k_2(l^2+m^2)=0$ will factorise, the one into real and the other into imaginary factors. This note provides a proof.

It is well known that $S=0$ represents a real line-pair if $D=0$ and $\mathcal{C}<0$. Since $\mathcal{B}\mathcal{C}-\mathcal{F}^2=aD=0$, it follows that $\mathcal{B}<0$, that is, $ac-g^2<0$. Thus the condition that

$$\phi+k_1(l^2+m^2)=0$$

represents real points is

$$(\mathcal{A}+k_1)\mathcal{C}-\mathcal{G}^2<0,$$

which can be written

$$bD+k_1\mathcal{C}<0\,.$$

Now we calculate

$$(bD+k_1C)(bD+k_2C)=b^2D^2+(k_1+k_2)bCD+k_1k_2C^2=-h^2D^2.$$

Therefore one of the factors $bD+k_1\mathcal{C}$, $bD+k_2\mathcal{C}$ is positive while the other is negative. This completes the proof. B. SPAIN.

2418. *A perpetual calendar.*

The following very simple rules for a perpetual calendar covering the first 2000 years of the Christian era have not, so far as I know, been previously given.

1. To the last two digits of the year add their fourth part, omitting fractions.
2. Subtract 1 if the date is on or before February 29 in a leap year.
3. If the year is before 1900, add 2: if before 1800 add an extra 2; if before 1700 add an extra 1 and so on for all earlier centuries.

4. If the day is on or before 2nd September, 1752, add 4.
5. Add a constant for the month of the year according to the following table :

January 1	April 0	July 0	October 1
February 4	May 2	August 3	November 4
March 4	June 5	September 6	December 6

6. Add the date in the month.
7. Divide by 7 and take the remainder. Calculate the day of the week from the remainder, starting with Sunday = 1.

A. D. DENTON.

2419. *A pretty series.*

Readers may care to devise their own proof of the following curious and somewhat pleasing result :

$$\sum_{r=1}^{\infty} \binom{(3r+1)}{r} x^r = \tfrac{1}{2}, \text{ when } x = 2/27 .$$

D. F. FERGUSON.

2420. *On alternating series* (*Note* 2279).

Professor Oppenheim's note 2279 (May, 1952) concluded with the question : " It would be interesting to know whether the convergence of S_1 and S_p $(p>1)$ involves the convergence of S_q for $q>p$." [The definition of S_q is

$$S_q = \sum_{1}^{\infty} (-1)^{n-1} u_n^q,$$

where u_n is positive and tends to zero, but not steadily.]

We are now able to answer this question in the negative and in fact to include Professor Oppenheim's results in the more general statement that the convergence for particular values of q does not imply convergence for any other values.

In the case where $q=1$ and $q=p$ give convergence, an example of a series which diverges for all other q may be constructed by taking

$$u_{6k-l} = a_l/\log k \qquad (a_l \geqslant 0,\ k = 2, 3, \ldots),$$

where

$$\left.\begin{aligned} a_{11} + a_9 + a_7 &= a_{10} + a_8 + a_6 \\ \text{and} \qquad a_{11}^p + a_9^p + a_7^p &= a_{10}^p + a_8^p + a_6^p \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

Then S_1 and S_p both converge to zero, and, provided that the a's are suitably chosen, S_q converges for no other value of q.

Theoretically, the solution of equations (I) may be regarded as selections of pairs of points of intersection of the plane $x+y+z=$ constant with the surface $x^p+y^p+z^p=$ constant. In practice, examples for which p and the a's are integral are furnished by the considerable literature on multigrade equations (see, for example, L. E. Dickson, *History of the theory of numbers* (New York, 1934), II, ch. XXIV).

For example, if

$$u_{24k-l} = a_l/\log k \qquad (k = 2, 3, \ldots),$$

where $a_l = 1, 0, 1, 2, 1, 2, 1, 2, 3, 2, 4, 2, 5, 7, 6, 7, 8, 7, 8, 7, 8, 7, 8, 9$ for $l = 47, 46, \ldots, 24$ respectively, then S_q converges for $q = 1, 2, 3, 4$ and 5 and presumably for no other values of q.

The triviality of the convergence to zero in the above examples may be removed by interspersing the u's with terms from any convergent series of positive terms ; for example, in the last example put $u_{24k-46} = 1/k^2$ instead of zero ($k = 2, 3, \ldots$).

Examples with p rational reduce to the integral case by the solution of the appropriate multigrades ; for instance, if

$$u_{12k-l} = a_l/\log k \qquad (k = 2, 3, \ldots),$$

where $a_l = 9, 100, 9, 100, 36, 144, 1, 1$ for $l = 23, 22, 21, 20, 19, 17, 15, 13$ respectively, and

$$u_n = 1/n^2 \qquad (n = 12k - 18, 12k - 16, 12k - 14, 12k - 12, k = 2, 3, \ldots),$$

then S_q converges for $q = 1$ and $\frac{3}{2}$ only.

Examples with p irrational would be more laborious to compute and less elegant in final appearance. RICHARD K. GUY.

2421. *Two problems.*

1. A general solution in integers of the equations

$$\left.\begin{aligned} a^2 + b^2 + c^2 &= d^2 + e^2 + f^2, \\ a^4 + b^4 + c^4 &= d^4 + e^4 + f^4. \end{aligned}\right\}$$

Example : $a = 15$, $b = 19$, $c = 31$, $d = 9$, $e = 25$, $f = 29$.

2. A general solution in integers of the equations

$$\left.\begin{aligned} g_1^2 + g_2^2 + g_3^2 &= 3(h_1^4 + h_2^4 + h_3^4) = 6k, \\ g_1^4 + g_2^4 + g_3^4 &= 9(h_1^8 + h_2^8 + h_3^8) = 18k^2, \end{aligned}\right\}$$

where $k = (h_3^2 - h_1 h_2)(h_3^2 + h_1 h_2)$.

Example : $g_1 = 7$, $g_2 = 34$, $g_3 = 41$, $h_1 = 3$, $h_2 = 4$, $h_3 = 5$, $k = 481$.

A. MOESSNER.

2422. *Note on $f(D)\{e^{ax}V(x)\}$.*

The following method of obtaining the usual result

$$D^n\{e^{ax}V\} = e^{ax} \,.\, (D + a)^n V$$

avoids the application of Leibnitz' theorem.

Assuming the above result for n, we have

$$\begin{aligned} D^{n+1}\{e^{ax}V\} &= D[D^n\{e^{ax}V\}] \\ &= D[e^{ax} \,.\, (D + a)^n V] \\ &= ae^{ax} \,.\, (D + a)^n V + e^{ax} \,.\, D(D + a)^n V \\ &= e^{ax}[a(D + a)^n V + D(D + a)^n V] \\ &= e^{ax} \,.\, (D + a)^{n+1} V. \end{aligned}$$

But the assumed result is true for $n = 1$ and hence by induction the general result is established. E. K. GREATRIX.

2423. *A note on the null plane.*

The existence and uniqueness of the null plane of a wrench is made clear by the following vector method.

Let the wrench $(\mathbf{F}, \boldsymbol{\Gamma})$ be equivalent to two forces, $\mathbf{F}_1$ through a given point 0, and $\mathbf{F}_2$. Then, if $\mathbf{r}$, $\mathbf{R}$ be the position vectors of points on the wrench and

on $\mathbf{F}_2$ respectively, relative to O, and such that $\mathbf{r}\boldsymbol{\Gamma} = 0$, then, taking moments about O,

$$\mathbf{R} \times \mathbf{F}_2 = \boldsymbol{\Gamma} + \mathbf{r} \times \mathbf{F}.$$

The right-hand side is a fixed vector and hence $\mathbf{R}$ and $\mathbf{F}_2$ lie in a fixed plane perpendicular to this vector. Any line, in the plane, through O, cuts $\mathbf{F}_1$ and $\mathbf{F}_2$, and is a null line ; hence the plane is the null plane of the system through O. Its equation is clearly

$$\boldsymbol{\rho}[\mathbf{r} \times \mathbf{F} + \boldsymbol{\Gamma}] = 0,$$

where $\boldsymbol{\rho}$ is the position vector relative to O, of a current point in the plane.

S. DIXON.

2424. *Light transmitted through a pile of parallel plates* (*Note* 2269).

Suppose that we have a set of parallel plates on which light of intensity I falls normally. And suppose that, when light falls normally on the kth plate in either direction, a fraction t_k of the light is transmitted and a fraction r_k reflected, where $t_k + r_k = 1$, so that no light is absorbed. Then we can prove by induction that the set of n plates transmits light of intensity $T_n I$, where

$$(1/T_n) = (1/t_1) + (1/t_2) + \ldots + (1/t_n) - n + 1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

which is obviously true when $n = 1$.

Since no plate absorbs light, the set of n plates reflects light of intensity $R_n I$, where $T_n + R_n = 1$. Moreover, assuming that equation (i) is valid for light incident normally in one direction, it remains true when the light is incident normally in the reverse direction, since (i) is unaltered if $t_1, t_2, \ldots, t_n$ are written in the reverse order.

Assuming then that equation (i) is correct, suppose that one more parallel plate is added to the set and that light of intensity I is incident normally. Light $T_n I$ passes through the n plates and falls on the $(n+1)$th plate. Of this, $t_{n+1} T_n I$ is transmitted at once. The remaining $r_{n+1} T_n I$ is reflected back to the set of n plates and $r_{n+1} R_n T_n I$ is reflected there to be sent back to the $(n+1)$th plate, where again part is transmitted and the rest reflected back to the n-plate set, and so on. It is readily seen that at the successive transmissions through the $(n+1)$th plate light passes of intensities

$$t_{n+1} T_n I, \qquad t_{n+1} r_{n+1} R_n T_n I, \qquad t_{n+1} r^2{}_{n+1} R_n{}^2 T_n I, \ldots$$

in all $$t_{n+1} T_n I / (1 - r_{n+1} R_n).$$

But this is equal to $T_{n+1} I$. Therefore

$$T_{n+1}(1 - r_{n+1} R_n) = t_{n+1} T_n.$$

Since $R_n = 1 - T_n$, $r_{n+1} = 1 - t_{n+1}$, we have

$$(1/T_{n+1}) = (1/T_n) + (1/t_{n+1}) - 1.$$

It follows that, if equation (i) is correct, it remains valid when n is replaced by $n+1$, so that the proof by induction can be completed.

If $t_1 = t_2 = \ldots = t_n = \frac{1}{2}$, equation (i) becomes $T_n = 1/(n+1)$, as proved by A. J. Moakes, (XXXVI, February 1952, p. 49).

H. SIMPSON.

2425. *A note on convergence.*

It is the practice in textbooks (my own included) to make quite heavy weather of the proof that $x^n \to 0$ in the interval $-1 < x < 1$, and the crossing for nx^n is even rougher. The usual argument writes $1/|x| = 1 + \delta$, $\delta > 0$, and then appeals to the inequality $(1+\delta)^n \geqslant 1 + n\delta$ to obtain $|x|^n < 1/n\delta$; for nx^n we use the next term in the binomial to give $n|x|^n < 2/(n-1)\delta$. In fact, we

do not need to look beyond the geometric series itself to find all that we require. For, if $0<x<1$, then

$$nx^n<x+x^2+\ldots+x^n=(x-x^{n+1})/(1-x)<x/(1-x). \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Whence, if $0<x<X$, taking x/X for x in (i), we have

$$nx^n<\{x/(X-x)\}X^n. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

From (i) it follows that $x^n\to 0$, and thence from (ii) that $nx^n\to 0$. Multiplying (ii) by n^k, we see that $n^{k+1}x^n\to 0$ if $n^kx^n\to 0$ (for $0<x<1$) which completes an inductive proof that $n^rx^n\to 0$ for any r. R. L. GOODSTEIN.

2426. *An oscillation problem involving elliptic integrals of the first and second kinds.*

1. *Introduction.* The problem is to find the *time* average of the string tension in a loss-free simple pendulum. Let m be the mass of the bob and l the equivalent length of the pendulum.* Then for oscillation *in vacuo* without loss, the differential equation is

$$ml\frac{d^2\theta}{dt^2}+mg\sin\theta=0, \qquad (|\theta|<\pi/2), \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(1)}$$

θ being the angular displacement at time t. The string tension T is the sum of the centrifugal force $ml\dot\theta^2$ and $mg\cos\theta$, the latter being the weight mg resolved along the string. Thus

$$T=m(l\dot\theta^2+g\cos\theta). \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(2)}$$

With $u=\dot\theta=d\theta/dt$, (1) becomes

$$u\,du/d\theta+(g/l)\sin\theta=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(3)}$$

so

$$\int u\,du+(g/l)\int\sin\theta\,d\theta=C_1, \text{ a constant}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(4)}$$

and, therefore,

$$\dot\theta^2-2(g/l)\cos\theta=C. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(5)}$$

At the extremity of a swing to the right, $\dot\theta=0$, and we take $t=0$, $\theta=\psi$. Substituting into (5) gives

$$C=-2(g/l)\cos\psi,$$

so

$$\dot\theta^2=2(g/l)(\cos\theta-\cos\psi)=4(g/l)(\sin^2\psi/2-\sin^2\theta/2), \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(6)}$$

and

$$\cos\theta=(l/2g)\dot\theta^2+\cos\psi. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(7)}$$

Using (7) in (2), we obtain

$$T=(3ml\dot\theta^2/2)+mg\cos\psi, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(8)}$$

so the time average of the tension over a quarter period τ, during which θ varies from ψ to 0, is

$$T_{av}=(3ml/2\tau)\int_0^\tau\dot\theta^2\,dt+mg\cos\psi. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(9)}$$

2. *Evaluation of integral in* (9) § 1. From (6) § 1, we have

$$\int_0^\tau\dot\theta^2dt=\int_0^\psi\dot\theta d\theta=2(g/l)^{1/2}\int_0^\psi(\sin^2\psi/2-\sin^2\theta/2)^{1/2}d\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(1)}$$

* It is assumed that the radius of gyration of the bob about a horizontal axis through the c.g. is small compared with the length l_1 of the string, *i.e.* l and l_1 are almost equal. Also that the string is inextensible and its mass negligible compared with that of the bob.

Taking $k = \sin \psi/2$, $k \sin \phi = \sin \theta/2$, we get

$$d\theta = 2k(1 - \sin^2 \phi)^{1/2}\, d\phi/(1 - k^2 \sin^2 \phi)^{1/2},$$

and

$$(\sin^2 \psi/2 - \sin^2 \theta/2) = k(1 - \sin^2 \phi)^{1/2}.$$

Substituting into the r.h.s. of (1) and noting that when $\theta = \psi$, $\phi = \pi/2$, we obtain

$$\int_0^\tau \dot{\theta}^2\, dt = 4k^2(g/l)^{1/2} \int_0^{\pi/2} (1 - \sin^2 \phi)\, d\phi/(1 - k^2 \sin^2 \phi)^{1/2} \quad \ldots\ldots(2)$$

$$= 4(g/l)^{1/2} \int \left\{ \frac{(k^2 - 1)}{(1 - k^2 \sin^2 \phi)^{1/2}} + (1 - k^2 \sin^2 \phi)^{1/2} \right\} d\phi \quad \ldots\ldots(3)$$

$$= 4(g/l)^{1/2}[(k^2 - 1)F(k, \pi/2) + E(k, \pi/2)]. \quad \ldots\ldots(4)$$

In (4), F and E are complete elliptic integrals of the first and second kinds, respectively, defined by

$$F(k, \pi/2) = \int_0^{\pi/2} d\phi/(1 - k^2 \sin^2 \phi)^{1/2} \quad \text{and}$$

$$E(k, \pi/2) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{1/2}\, d\phi, \quad \ldots\ldots(5)$$

with $k^2 < 1$.

By aid of the substitutions below (1), and the fact that $\dot{\theta} < 0$ in $0 < \theta < \psi$, (6) § 1 may be written

$$\int_0^\tau dt = (l/g)^{1/2} \int_0^{\pi/2} d\phi/(1 - k^2 \sin^2 \phi)^{1/2}. \quad \ldots\ldots(6)$$

Thus we obtain the well-known formula for the quarter period, namely,

$$\tau = (l/g)^{1/2} F(k, \pi/2). \quad \ldots\ldots(7)$$

Using (7), (4) in (9) § 1 leads to

$$T_{av} = (6mg/F)\{E + (k^2 - 1)F\} + mg \cos \psi. \quad \ldots\ldots(8)$$

Writing $\cos \psi = (1 - 2k^2)$, we obtain the exact expression

$$T_{av} = mg\{6(E/F - 1) + 4k^2 + 1\}. \quad \ldots\ldots(9)$$

3. *Approximation to* (9) § 2. If the amplitude of swing ψ is small enough, $k = \sin \psi/2 \simeq \psi/2$. Then, by (5) § 2,

$$F \simeq \int_0^{\pi/2} d\phi(1 + \tfrac{1}{2}k^2 \sin^2 \phi) = (\pi/2)(1 + k^2/4), \quad \ldots\ldots(1)$$

and

$$E \simeq \int_0^{\pi/2} d\phi(1 - \tfrac{1}{2}k^2 \sin^2 \phi) = (\pi/2)(1 - k^2/4), \quad \ldots\ldots(2)$$

so

$$(E/F - 1) \simeq -k^2/2. \quad \ldots\ldots(3)$$

Inserting (3) into (9) § 2 yields the approximate result

$$T_{av} \simeq mg(1 + k^2) \simeq mg(1 + \psi^2/4). \quad \ldots\ldots(4)$$

N. W. McLachlan.

1780. Is it an accident that fundamental principles are most often evolved as the result of experiments or specific problems? Is it not probable that the most efficient way to learn is the way the human race has learned—first from specific problem or experiment to general principle and then to new problems? Is not too much effort spent in trying to reverse the process rather than to improve it?—K. B. McEachron, "Why Options Anyway", *Electrical Engineering* (Journal of the Amer. Inst. of Elec. Engrs.), p. 305. [Per Mr. C. C. Puckette.]

REVIEWS

The Printing of Mathematics. By T. W. CHAUNDY, P. R. BARRETT and CHARLES BATEY. Pp. ix, 105. 15s. 1954. (Geoffrey Cumberlege, Oxford University Press.)

This Oxford team, consisting of the University Reader in Mathematics, who is also an editor of the *Quarterly Journal of Mathematics*, the Mathematical Reader at the Press, and the Printer to the University, gives us the eagerly awaited volume of mathematical typography so badly needed. Young authors should be compelled to read it, possibly as a " set book " for the Tripos ; the experienced author who can say of its admonitions, " All these have I kept from my youth up ", may glow with pride, if he exists.

There are three main sections : the mechanics of mathematical printing ; recommendations to mathematical authors ; the Oxford rules for the composition of mathematics. In the first section, a marvel of lucid compression, Mr. Batey describes the printing processes which transform an author's manuscript into the finished specimen of typographical craftsmanship. In the second, Dr. Chaundy and Mr. Barrett give us the whole duty of the author, based on the principle that " mathematicians who are writing in English are asked not to forget the dignity and traditions of the language " and that " what they write purports to be English prose ". This principle is applied to every detail connected with the preparation of manuscript, from the proper use of the word " function-theory " to the organisation of brackets. Moreover, the simple but neglected fact that true communication of thought from writer to reader depends on the skill of the printer is shown to demand full co-operation between author and printer, so that, for instance, the writer should know why it is better to print $\frac{1}{2}\pi$ rather than $\pi/2$ and how proof corrections should be indicated. The third section reprints the house rules for composition of mathematics at the Oxford Press, hitherto available only to the Press authors, editors and printers. There are three short appendixes, on legible handwriting, on type specimens and on abbreviations.

The authors ask for constructive criticisms or suggestions. I offer them two.

The colon is scurvily treated in a short paragraph which says : " The *colon* is rather precious in its use. . . . It has been usefully said that a colon is appropriate where a letter-writer would have put a dash." Are the authors charging the Authorised Version or Gibbon with preciosity, or are they merely refusing here to make a stand against the steady impoverishment of the English language, whereby it is rapidly losing its fine shades of meaning and its delicate system of punctuation? The separation of an introductory clause from a string of subordinate clauses is conveniently effected by a colon, when the subsidiary clauses are themselves separated by semi-colons, as in the first sentence of the second paragraph of this review. This usage, to be found in the best English prose, is particularly effective in mathematical prose. Further, the " useful " remark in the second part of the quotation is surely false. I took down a number of volumes of English letter-writers from my shelves and substituted colons for dashes in 50 instances taken at random. Only once did the altered sentence remain intelligible. I suggest that the authors should rewrite this section, replacing the " useful " remark by a short exposition of the correct doctrine of the colon.

Secondly, experiments on symbols for the real and imaginary parts of a complex number have led the authors to recommend " re z " and " im z ". They may be pleading for a consistent use of lower case Roman in functional signs, since these notations are offered in a table of standardised functional symbols such as $\sin x$, $\log x$, $\mathrm{cn}\, x$. Now the real and imaginary parts of a complex z are certainly functions of z, but they are hardly on the same footing

as the classical elementary functions and a more distinctive mode of calling attention to a decomposition of the complex number is preferable. For blackboard work, the distinction between Roman and italic can not be maintained; and one letter is better than two. For these reasons, a script letter is convenient on paper or blackboard, and the beautiful upper case script, in boldface variant, would be ideal for typographical purposes but for technical objections to undue use of this fount. However, the Press's Fraktur series, which is practically the old and fairly plain German fount, not the rococo Gothic, would be suitable, namely $\mathfrak{R}(z)$, $\mathfrak{J}(z)$.

A magnificent book, informative, stimulating, and superbly produced, it should do much to improve the standard of mathematical manuscripts. I hope it may also interest many readers in a craft in which Great Britain may have peers but certainly no superior. T. A. A. B.

Topologie. I. By C. KURATOWSKI. 3rd edition. Pp. xi, 450. $7.50. 1952. Monografje Matematyczne, 20. (Warsaw)

The third edition of this standard work is a French translation of the second edition, which was published in 1948. The author's reasons for issuing a second edition were, first, the exhaustion of the first edition, and, second, the imminent publication of the long-awaited Volume II. The main modifications in the second edition were motivated first by a desire to complete those parts of the subject which find no place in Volume II, and, second, to provide an adequate foundation for the material of Volume II (though the author claims that the reader familiar with the first edition of Volume I will be able to proceed to Volume II without noticing any violent discontinuity). In view of the well-deserved reputation of this outstanding textbook of point-set topology, this review will be devoted largely to outlining the modifications.

To achieve the first object mentioned above, the sections dealing with Borel sets, Baire functions (that is, functions $f : X \to Y$ such that, if F is a closed set in Y, $f^{-1}(F)$ has the Baire property), and the theory of projective sets, due to Souslin, Lusin and Sierpinski, have been greatly enlarged. The effect is to render much of the material of Chapter III difficult to understand on first reading, but the author has achieved his object of providing a definitive treatment of these topics.

The intended discussion of mapping-spaces in Volume II has led the author to insert in this volume some additional theorems on compact L^*-spaces (that is, compact spaces in which the notion of limit may reasonably be defined). However, the most significant clue to the projected contents of Volume II is provided by the inclusion of an entirely new section (§23) on simplicial complexes and polytopes. It should be mentioned here that a complex is not subjected by Kuratowski to the condition that it shall contain the faces of all its simplexes; this condition characterises, in the author's convention, a *closed* complex. The new section contains much the same as paragraphs 1, 2, 3 and 10 of Pontryagin's *Foundations of combinatorial topology*†, though the latter is easier reading for the English student. That is to say, the section contains a treatment of the topological dimension of a simplex, based on Sperner's lemma, a proof and applications of Brouwer's fixed-point theorem, the definition of the nerve of a system of sets (in particular, of an open covering of a space) and its geometrical realisation, and a theorem on approximating to continuous mappings of metric spaces into bounded subsets of euclidean space. The aim of this part of the theory is to prepare for the theorem, to be proved in Volume II, on the embeddability of a separable metric space of n dimensions in euclidean space of $(2n + 1)$ dimensions. This theorem is proved,

† Graylock Press, Rochester, N.Y. (1952); review in *Gazette*, XXXVII, 311.

for compact spaces, in paragraph 3 of Pontryagin (p. 22). Section 23 also introduces the ideas of infinite complexes and extensions of mappings.

It is curious that no use is made of such standard terms as *subdivision*, *star* (although the ideas they express are constantly in service); other fundamental ideas (*e.g.*, simplicial map) are omitted, but this may well be dictated by the intended contents of Volume II.

We now briefly review Volume I as a whole. After an indigestible introduction, the present edition is divided, as was the first edition, into three chapters, dealing respectively with fundamental notions, separable metric spaces, and complete spaces. It is, incidentally, remarkable that the contents of Sierpinski's *General topology** are very similar and very similarly arranged, though the latter work does not carry its subject-matter so far. In the first chapter of the present work, not only are the fundamental ideas of point-set topology defined (*e.g.*, closure mapping, interior, frontier), but the calculus of the associated operations ($X \to \bar{X}$, $X \to f(X)$, $X \to \mathrm{Int}\ X$, $X \to \mathrm{Fr}\ X$) is systematically developed. This makes for great fluency in many proofs, but tends to render them somewhat formal and non-intuitive in appearance.

The index seems to the reviewer to be inadequate; in a work of this kind, so largely consulted for reference, it should not be assumed that a reader studying a given paragraph retains his familiarity with the terms and symbols introduced in the preceding paragraphs. The choice of X' for the derived set of X is unfortunate typographically (for example, it leads "by default" on p. 45, top, to the erroneous statement $X' - Y' = X - Y$), and there are a few misprints surviving from previous editions (*e.g.*, p. 46, line 9 should read "Il vient $G = 1 - F$ ..."), but these are minor criticisms

We must be grateful to Professor Kuratowski for again making this work available, in a language we understand, and we look forward to reading Volume II (now, apparently, published) with keen anticipation.

P. J. HILTON

Introduction to the Foundations of Mathematics. By R. L. WILDER. Pp. xiv, 305. 46s. 1952. (John Wiley, New York; Chapman and Hall)

The author describes his function in this work as that of "reporter and expositor" (p. 188). To set forth rival points of view, to "provoke" his readers, to induce personal reflection, criticism, and, in final analysis, a search of the relevant sources themselves, such are his express aims. All this he does, throughout eleven chapters, somewhat as a Judge sums up at a trial, withholding his own opinion and stating the evidence presented by each side. In the twelfth and last chapter (of 20 pages), which is in the nature of an Appendix, he deliberately relaxes this restraint, but with the same object in view, of provoking discussion and, in this case, eventually opposition; at the same time, his beliefs are mildly and even tentatively framed, and take a line so often adopted in a wide range of contemporary writings as to constitute a species of modern orthodoxy: namely that of attributing effective responsibility to the social background rather than to individual initiative.

On the constructive side, the work chiefly expounds the axiomatic approach in mathematics, with particular reference to concepts on which "modern mathematics" relies. By "modern mathematics" is meant the apparatus of Analysis, Algebra and Geometry as taught in normal mathematics courses; and the specific concepts discussed are those that have in recent times been subjected to authoritative criticism and are in part still *sub judice*, above all the nature of axioms, the logic of proof, different kinds of infinities and num-

* University of Toronto Press (1952); review in *Gazette*, XXXVIII, 58.

bers, ordering and well-ordering, the abstract notions of set, group and operations, in fact all the topics that it is customary to collect under the title "Foundations of Mathematics", where little notice is usually taken of the relatively uncontroversial specialised ingredients of modern mathematics, *e.g.* limit theory and the geometry of sets of points and of special configurations, from which the axiomatic theories of sets and groups actually originated. In this respect, admittedly, the historical setting of Prof. Wilder's account is perhaps incomplete, but follows the fashions of the day, and the same may be said of the portrait of modern mathematics which his account provides. But in a new portrait an artist may seek, instead of greater completeness, a more vivid rendering of a typical feature. In the event, a far more living and human entity emerges as mathematics than is usually shown, unless to fellow-experts within some narrow field of work. This is largely due to the author's averseness, plentifully indicated, to instilling mathematics as "dogma", even at elementary stages. There is no deadly finality about the subject-matter presented. This is frankly a feat in a piece of "reporting", which, incidentally, contrives to give considerable information not hitherto generally accessible, in particular on Intuitionism and its bibliography.

An occasional slip, deliberate or otherwise, only enhances the humanity of the whole. One that may not be met with an efficacious incredulity outside Britain (surprising though the fact may seem to English readers) concerns the use of Euclid as a text-book in English schools. Even from these, it has long since disappeared. The task of substituting appropriate up-to-date logical exercises remains, a difficult task, which presupposes a belief in the logical capacities of a youthful mind such as effectually only Lewis Carroll, and his other self Dodgson, have systematically displayed. The same unquenchable belief, however, is apparent at the higher level in Professor Wilder's book and should not be impossible to extend downwards. It has a practical basis to which he himself testifies, and which explains the understanding with which he faces the difficulties of the inquirer, in numerous entertaining illustrations, and in the exercises and instructions for further reading. Behind lie 20 years of lecturing to "teachers, actuaries, statisticians and others who had specialised in undergraduate mathematics", and the encouragement of their enthusiastic response. There lies also, according to the author, the example of J. W. Young's "Fundamental Concepts of Algebra and Geometry" of 1911, and the plan, long since conceived, of bringing that work up to date.

R. C. H. YOUNG

A School Course in Mechanics. Part I. By A. J. BULL. Pp. viii, 156. 8s. 6d. 1952. (Cambridge University Press)

The author's preface states that he has tried to fulfil three requirements—namely, to write a continuous course suitable for all school purposes except for the final preparation of the mathematical scholar, to introduce absolute units early on and to use them wherever appropriate, and to bring the book into line with modern teaching practice, so that the calculus is freely used (but principally this applies to Part II). Part I embraces all that in the past has been associated with Additional Mathematics at School Certificate, and for that reason very little calculus is done in it.

The reviewer, having read the above remarks in the preface, found on examination of the book that in an attempt to provide a continuous course which was logically coherent, the author has unfortunately put large stumbling blocks in the way of the young beginner in mechanics. The book is an admirable sixth-form text for beginners in mechanics whose mathematical knowledge is otherwise up to about pass level in Additional Mathematics, but

its demands on the pupil starting his mechanics two years earlier are far too exacting.

Three references may be made to illustrate this point and to show the methods the author uses.

(i) An attempt is made to establish Newton's second law of motion in the form $P = ma$ by means of experiments with a Fletcher's trolley, in which it is not made clear how any force can be measured or how the results of the experiment lead to the conclusion $P = ma$.

(ii) The work on the parallelogram law (with which the statics course opens) leads straight into the calculation of the resultant of two forces by the cosine rule and a method for finding *general* components of a force using the sine rule. Only later is the special case of perpendicular components mentioned and no graphical work is done or suggested until the resultant of five forces at a point is found graphically. It is, however, only fair to point out that the author sets plenty of easy questions to be done graphically by the reader.

(iii) The chapters on " Work " and " Moments " each commence with the *general* definition of these quantities and the simpler special cases which are so common, and with which it is surely more natural to introduce these concepts, are derived from the general definitions.

The author is clearly anxious to show the theoretical connection between the various principles in the subject. Thus Varignon's theorem, that the sum of the moments of two forces about any point is equal to the moment of their resultant about that point, is first proved, and the principle of moments is deduced from it. This procedure, however admirable from a theoretical point of view, will create difficulties for the teacher who commences his statics with the law of the lever, and is best left for an advanced text-book. On the other hand no attempt is made to justify the principle of work for a machine, although the consideration of machines follows after work on energy and power.

Very little appeal is made to the reader's intuition or observation, in the discussion of fundamental principles, which is otherwise admirably clear. In worked examples much trouble has been taken in the selection of examples and in the setting out of the work. The selection of problems (mostly in sets at the end of each chapter) is excellent, but in many chapters there is a need for more easy " drill " examples.

It is to be hoped that in a later edition the suggestions for practical work will be considerably expanded.

The printing is excellent, type and diagrams being alike admirably clear and well spaced, and the pages bear a most attractive appearance.

Attention is drawn to the following details :

p. 4. The velocity on a time-distance curve is given by the gradient of the curve and *not* by tan ψ, for the latter depends on the scales chosen and is a pure number.

p. 46. Second paragraph, line 3, the third word is *of*, not *to*.

p. 65. The solution at the foot of the page is made much simpler if λ is found from tables and 10 tan $(30° - \lambda)$ is then obtained. The use of g for grams is somewhat confusing and unusual.

F. J. T.

Deuxième Colloque de Géométrie Algébrique. Pp. 244. 2625 fr. 1952. (Thone, Liège ; Masson, Paris).

The fourth Colloquium organised by " Le Centre belge de Recherches mathématiques " was, like the first, devoted to Algebraic Geometry, and was held at Liège in June, 1952. This volume contains the proceedings of the Colloquium. It contains thirteen addresses, delivered by geometers from half

a dozen countries. The addresses deal not only with the classical algebraic geometry of the Italian school, but also with the two main branches of mathematics which may be said to have grown out of the classical theory, namely, on the one hand, modern abstract algebraic geometry, and, on the other, the theory of complex manifolds. While a considerable number of new results are published in the addresses, a number consist of surveys of the branch of the subject in which the authors are interested, and these are likely to prove very valuable to a wide range of mathematicians, particularly when, as is sometimes the case, they are supplemented by excellent bibliographies.

W. V. D. H.

Daily Life Mathematics. II. By P. F. BURNS. Pp. 244. 8s. 1952. (Ginn)

Mr. Burns continues with his policy of dealing with centres of interest and of activities out of which mathematical ideas arise and are used as they are needed. To be fully effective, the activities which are suggested, and are often described in detail, should actually be carried out and these require a workshop rather than an ordinary classroom. In this way, the mathematics takes on a real meaning. The book starts with fitting of linoleum and upholstering of furniture. Isometric drawings and plans and elevations follow naturally. Percentages centre round such topics as house purchase and local rates. Bicycles and gears introduce the pupils to the mensuration of the circle and there are examples on the orbits of the Earth and other planets. The Plimsoll Marks and buoyancy treated experimentally follow a chapter on metric weights and measures. There is a chapter on practical surveying in which the construction of simple but useful instruments is described. Later, there is a chapter on Earth Measurements. From this, Time, the Calendar and the construction of a sundial follow naturally. Earlier, there is a chapter on Pythagoras, and the method of calculating square roots is developed on the lines described by Sir Percy Nunn. As in Book I, there is a set of routine practice exercises at the end. The author makes no attempt to deal with Algebra and ideas connected with this subject, so that some teachers would want to supplement this book when they deal with their best streams. For other streams, teachers will find this book valuable and fruitful. S. I.

Mathematics Today. III. Largest and Smallest. By E. E. BIGGS and H. E. VIDAL. Pp. 193. 5s. 6d. 1952. (Ginn)

Part III is in two chapters. The first, of 84 pages, deals with gradients. It starts with the gradient of a straight line and the idea is later expanded to include simple cases of x^n and easy polynomials. The development occupies many pages and, occasionally, heavy weather is made of it when a more concise treatment would produce an easier and quicker comprehension. Some 40 pages are devoted to gradients and the general rule for x^n is not stated at the end. The symbol $\frac{dy}{dx}$ is given but without mention of δy, δx and the ratio $\frac{\delta y}{\delta x}$. These occur first on p. 105 towards the end of the course. The first chapter gives some 40 pages to applications of approximations, which is welcome, velocities, accelerations and maxima and minima.

The second chapter of 56 pages deals with integrations, areas and volumes of solids of revolution. There are 29 test papers, 12 being short and 17 of full examination length. S. I.

Linear Algebra and Matrix Theory. By ROBERT R. STOLL. Pp. xv, 272. 51s. 1952. (McGraw-Hill)

This textbook gives an account of matrix theory against the background of modern algebra. It is intended for both pure and applied mathematicians,

but does not deal with the applications ; it is, indeed, written from the pure mathematician's point of view. The author makes considerable, and judicious, use of the abstract method. He invariably gives a careful account of the concrete situations as well as the abstract generalisation. The development of an abstract theory is necessarily rather formal, and many students will find the text difficult in places. Nevertheless, the book is well written and skilfully arranged. The methods are carefully chosen so as to bring out the true meaning of the theory and the author's explanations, both of general purpose and details, are full and clear. The text is illustrated by worked examples, and there are problems at the end of each section.

Chapters I–V deal with linear equations, vector spaces, basic operations on matrices, determinants, quadratic and Hermitian forms. The account of linear equations, following Hasse's " Höhere Algebra ", is concise but excellent. The theory of vector spaces is carried as far as factor spaces (though this last concept is only used in the later parts of the book). Determinants are treated postulationally. Here the development is rather sophisticated, *e.g.*, the existence proof comes after most of the familiar properties have been established. The technique of elementary transformations on rows and columns of a matrix plays an important part in these early chapters. It is used, for example, in rather a novel way to derive the standard theorems about rank, definiteness, and principal minors, of a symmetric matrix.

The last three chapters, on similarity and unitary similarity, are perhaps the most interesting of the book. Groups and rings are introduced in the preparatory Chapter VI. The treatment of similarity in Chapter VII is in essence, though not in form, that of van der Waerden's " Moderne Algebra ". Let the matrix A act on the vector space V. The author defines, in an evident way, " A-generating systems " of V. For each such system $(\alpha_1, \ldots, \alpha_r)$ we have " relation rows " $(f_1(\lambda), \ldots, f_r(\lambda))$ such that $\sum_i f_i(A)\alpha_i = 0$, and by putting together enough relation rows to span the space formed by all of them we get " complete relation matrices ". The crucial point is that similarity reduction of A is mirrored by equivalent reduction $X \to PXQ$ of a complete relation matrix ; in this way we are led naturally to the elementary divisor theory. To tie this up with the usual method, viz., equivalent reduction of $\lambda I - A$, the author remarks that $\lambda I - A$ is itself a complete relation matrix with respect to a natural basis of V. The method is subtle, but has the great advantage of showing why the elementary divisors come in ; it also explains the companion matrices in the canonical form, which in some accounts simply appear out of the blue.

The final chapter gives a stimulating account of finite dimensional unitary spaces, culminating in the unitary similarity theory of normal matrices. The author's " geometrical " approach has decided advantages of clarity, though he perhaps carries his method too far in the rather elaborate proof of the existence of the adjoint. The fact is evident, since the space is finite dimensional, from the matrix expression $\bar{x}'Kx$ of the fundamental Hermitian form.

In conclusion, this book may be strongly recommended for those who wish for a thorough understanding of matrix theory. Only one serious error was noted : the statement (p. 27) that all solutions of a system of linear equations lie in the field of the coefficients.

G. E. WALL.

Complex Analysis. By L. V. AHLFORS. Pp. xi, 247. 42s. 6d. 1953. International series in Pure and Applied Mathematics (McGraw-Hill)

The English reader is well served with books on the complex variable, Copson's volume being particularly noteworthy. This new work by Ahlfors,

once a pupil of Lindelöf's and now Professor at Harvard, is nevertheless well worth adding to the teacher's shelf, because of its re-distribution of emphasis, its variations in method, and some novelty of content. There are six chapters : Complex numbers ; complex functions ; complex integration ; infinite sequences ; the Dirichlet problem ; multiple-valued functions.

The concept of conformal mapping is strongly emphasised in the first two chapters. Ahlfors says that he learned from Lindelöf to place this topic before contour integration, and most teachers will agree with him, particularly if they are concerned with pupils who will have to use the complex variable in scientific or technical applications. With a few simple ideas, a deep penetration and a remarkably thorough insight into the theory of functions of a complex variable can be obtained before any attack on Cauchy's theorem and the calculus of residues is made.

In dealing with the circle of ideas surrounding Cauchy's theorem, much use is made of the simpler terms and concepts of modern topology in order to separate the difficulties which are analytical from those which are essentially topological. To talk of " cycles homologous to zero " is perhaps only a change of phrase, but it makes for precision of statement, and the systematic use, following Artin, of the " winding number " of a curve γ with respect to a point a, defined by

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

simplifies and clarifies enunciations and proofs.

The first four chapters are classical in content. But the fifth chapter, though simple enough to read, deals with topics to which an elementary introduction has not hitherto been easy to find. Subharmonic functions are studied after the work on harmonic functions, so that Perron's simple method for the solution of Dirichlet's problem can be expounded, and an introduction to the mapping problem for multiply-connected regions is given. In the final chapter, introductory ideas about multiple-valued and algebraic functions are set forth, and applied very effectively to the theory of linear differential equations in the complex domain.

Altogether, a book to be warmly recommended as a clear, precise, and forward-looking introduction to the theory of functions of a complex variable. Having said this with, I hope, due emphasis, two small grumbles may be permitted. The author's uniform style sometimes masks the key-result, and in a revision he might consider some re-wording so as to concentrate attention at the critical points. Secondly, some suggestions for further reading would be valuable. T. A. A. B.

Servomechanism Analysis. By G. J. THALER and R. G. BROWN. Pp. xii, 414. 64s. 1953. (McGraw-Hill)

Although some years ago the subject of servomechanisms had been well covered by numerous papers in technical journals and privately circulated reports, few text books had appeared. Various writers have however come forward recently to remedy this deficiency.

The topics covered in this latest book include the usual processes of transient analysis and frequency response, with detailed accounts of their application and interpretations treated on conventional lines. Of particular interest are a proof of the Nyquist stability test and an alternative method of examining stability, called the root locus method. Most authors seem to shy at the former. It is true that engineers are not necessarily interested in proofs which are essentially mathematical exercises, but the proof in this case (which

is not difficult) has the merit of showing clearly the restrictions to which the rule is subject, and in particular the modification necessary if the open loop system is unstable.

The root locus method is interesting. Briefly it is as follows. If $F(z)$ is the open loop transfer function, stability of the closed loop system depends on the location in the z-plane of the roots of the equation $F(z) = -1$. This is equivalent to $\arg F(z) = \pi$, $|F(z)| = 1$. Since $F(z)$ is usually given in factorised form, its zeros and poles can be marked on the z-plane directly. It is then quite easy to infer the general shape of the locus $\arg F(z) = \pi$, and to produce an approximate graph. This is called the root locus. The required roots are then located by trial and error methods. This process not only lightens the arithmetical labour of solving the characteristic equation, but, what is even more important, it can also be used to suggest modifications in order to improve performance.

This book is soundly and carefully written, even though the explanation of some of the mathematical theory tends to be long winded. It provides a rather more comprehensive account of the subject than some of the other books that have appeared recently. Even though the authors make no claim to originality of material or treatment, they have made a worthy contribution to the literature of the subject.

B. M. BROWN

Differential and Integral Calculus. By P. FRANKLIN. Pp. xi, 641. 48s. 1953. (McGraw-Hill)

The author is Professor of Mathematics at the Massuchusetts Institute of Technology and this is a text-book "intended for a year's course for engineering and science majors". As may be judged from the number of pages the student required to study the whole of the material provided would have a hard year's work, but it is implied that selection would be made according to the needs of the course and the calibre of the student.

The book is written on several levels. In addition to the main text there are two sorts of section set in small type. Those marked R are for revision of trigonometry, algebra and analytical geometry, while others marked with an asterisk give the more rigorous proofs or treat topics not regarded as essential. Further differences of level are achieved by repetition, the most obvious example being a dual treatment of integration, applied first with simple algebraic functions to determinations of area and fluid thrust and later defined as a limiting sum and treated systematically. Minor examples of the same method are the solution of some differential equations of the second order with constant coefficients at an early stage and again in the final chapter and some formulae of integration obtained first in logarithmic form and later in terms of inverse hyperbolic functions.

There are some difficulties as well as advantages in such a method of presentation. Quite often, for example, it is necessary to refer in the main text, or even in a revision section, to results obtained in a more difficult starred section. Teachers would be likely to disagree on the classification of material, especially as between main text and starred sections. Putting aside one's own prejudices in this respect it is only fair to say that the author has made a good job of it. The whole treatment is thorough and attention is paid to those details which experienced teachers recognise as causes of difficulty or misunderstanding.

The range covered has been partly indicated above. In addition to differentiation and integration of the usual range of functions, with applications to curves in cartesian and polar coordinates, plane motion of a particle, centres of gravity and pressure and moments of inertia, there are chapters on

convergence, Taylor's series, partial differentiation up to maxima and minima of a function of two variables and multiple integrals. There is an introduction to vector algebra, stopping short of the triple vector product. The last chapter, on differential equations, deals with the usual first order types, equations of higher order with constant coefficients and, by way of examples, the substitution $x = e^t$ for solving Cauchy's equation. Simultaneous and partial differential equations are not included. A standard of rigour higher than is usually found in text-books for technical students is often attained. For example, Duhamel's theorems are explained and used to justify approximations in a summand before passage to the limit as a definite integral.

There are plenty of worked examples and exercises for the student. A possible criticism of the latter is that they are mainly drill exercises; some more problems requiring the translation of physical data into mathematical form could with advantage be set to technical students. The omission of (p, r) equations with their application to curvature and orbits is perhaps to be regretted. Generally it is assumed that problems of dynamics are solved from force-acceleration equations, but engineering students should appreciate the double saving—of an integration and of the introduction of unnecessary forces—achieved by the use of energy and momentum equations.

In so large a book the following list of typographical errors noted amounts to very little. On p. 25 figs. 14, 15 have been placed in section 22 but belong to *R* 23; on p. 184, line 8, the author presumably means $(1+h)^{1/h}$, not $(1+1/h)^h$; a lower limit has been omitted in integrals on p. 300, equation (42) and p. 540, equation (9); the first index $(n-1)$ occurring in equation (43) on p. 449 should be $(n-2)$. C. G. P.

Practical Calculus. By C. I. Palmer and Claude E. Stout. Revised Second Edition. Pp. xx, 470. 51s. 1952. (McGraw-Hill)

This book aims to teach a student the essentials of the Calculus without other assistance. With this aim in view, all the explanations are very full, and in most cases, complete. The examples chosen—both worked and for exercise—are the best features of the book and fully justify the title-word "Practical". Other good features are the clarity of the diagrams and the printing, which is of the usual high American standard.

However, as a self-tutor, the book possesses several defects. Some of the material is over-specialised, for example, cusp maxima and minima are covered before the student has attempted ordinary practical max.-min. problems for himself. Two vital points which could be improved are the derivative of a power of a function where the method adopted is tedious, and the rather pathetic introduction to the derivative of logarithmic functions which states "The following derivation is perhaps as simple as any that can be given. If, however, the reader finds it too difficult to follow, he should remember that he can use the formula even though he cannot derive it." The complete chapter on Analytical Geometry could well have been omitted from a Calculus text-book costing 51 shillings.

As a reference book, for both lecturer and student, much of the book will be useful and would therefore be a useful addition to the mathematical section of Technical Libraries. D. M. Melluish

Practical Mathematics. By C. I. Palmer and S. F. Bibb. 4th Edition. Pp. xii, 769. $6. 1952. (McGraw-Hill)

The subject matter is equivalent to that of the first two years of the Ordinary National Certificate course. The book is intended for home study and consists of four parts; Arithmetic, Geometry, Algebra and Trigonometry.

The first contains useful chapters on fractions, percentage and proportion.

The examples are practical and well graded. Gallons and short tons may cause trouble in the Weights and Measures section.

Part II, Geometry, deals with the mensuration of the triangle, circle, cylinder, pyramids, frusta and the sphere. Many practical applications and approximate methods are discussed.

There are plenty of drill exercises in the section on Algebra. The chapter on graphical methods is rather short and the determination of experimental laws is omitted.

The layout of the section on trigonometry is considered to be unsuitable for most students. All six ratios, angles of any magnitude and the evaluation of formulae are tackled before the solution of triangles.

Although this book is not recommended as a textbook for National Certificate students it is worth the attention of adult students wishing to refresh their mathematics and should be available in Technical College libraries.

F. G. JEEPS

Algebra for College Students. By R. R. MIDDLEMISS. Pp. viii, 394. 32s. 1953. (McGraw-Hill)

English teachers who have complained of the bad grounding in algebra of students entering on a university course may find some comfort in considering the apparently worse plight of their brethren in adversity in U.S.A. In his preface Professor Middlemiss states that "many high school administrators, and university professors in the field of education, have questioned the value of algebra, essentially on the ground that its main purpose is to get phony answers to phony problems." This amounts to a severe criticism of the methods and text-books of the high schools, particularly if even those students who are to study mathematics at the university have not come to appreciate the value and importance of algebra.

The author starts at the beginning with positive integers and proceeds quite slowly with fractions, negative and irrational numbers. It is true that he is at pains to show how different types of numbers arise, a useful lesson, but he also finds it necessary to set exercises of a very simple character, including drill in the most elementary processes. The first twelve chapters, extending to p. 212, are essentially at O level, except that complex numbers are introduced on an empirical basis with quadratic equations. A good feature of this half of the book is the insistence on the fact that an equation is merely the hypothesis in a chain of reasoning and that unless the steps of the argument are reversible it is necessary to prove the converse proposition by substitution before asserting that the equation has been solved.

The second half of the book deals with mathematical induction, the binomial theorem (the infinite series and its validity are briefly stated), theory of equations, determinants, complex numbers, permutations and combinations and probability. The treatment is that of an easy first course in these topics; for instance, the work on theory of equations is mainly numerical and there is no mention of symmetric functions of the roots. The algebraic solution of the cubic equation is given. It is in this section that the only error noted occurs: on p. 283 coefficients a, b, c of the equation are replaced later by b, c, d.

C. G. P.

Differential Equations. By R. C. YATES. Pp. vii, 215. 32s. 1952. (McGraw-Hill)

We read on the wrapper that this book is designed to prepare students for work in modern engineering practice and it may be said at once that it has much to recommend it to the technical student. For the most part it is written in a bright and practical style; the reader, noticing in the index

" Parabolic reflector, 25 ", will find on that page a worked example beginning " A bucket of wet plaster of paris in fluid form is revolved about its axis . . ." or he may be pleased to learn on p. 116 that the vibration absorber just discussed has recently been incorporated successfully into an electric shaver. Less attractive as written English is such a sentence as " Ready references to trigonometry, analytics and calculus should have elbow room on the study desk " or the statement on the wrapper that the approach is " particularly unique ".

The ground covered constitutes a very useful course for an engineering student, including the usual range of ordinary differential equations together with the solution of the partial differential equations of heat-flow and vibrations (with two dependent variables only) by means of separation of variables and Fourier series. Legendre's and Bessel's equations are solved and the Gamma function introduced. At intervals through the book short chapters occur entitled " Summary and Review Problems ".

A few criticisms of details may be made : (i) In the discussion of the function $Me^{-at}\cos(bt+\phi)$ it is stated that the amplitudes $\pm Me^{-at}$ are maximum-minimum values of the function. (ii) Fig. 9.3 is somewhat misleading, having an axis OY positively upwards but a downward deflection marked y. (iii) In the problem of the free surface of rotating fluid (the wet plaster) centrifugal force is introduced and tangential forces balanced. The student might be perplexed at the apparent lack of balance of the normal forces, since no mention is made of fluid or atmospheric pressure.

No misprints have been detected and the lay-out and printing are very satisfactory. C. G. P.

1. Plane Geometry. A clear thinking approach. Pp. xi, 436. 27s. 6d.
2. Solid Geometry. A clear thinking approach. Pp. x, 198. 25s. 6d.
By L. H. SCHNELL and M. G. CRAWFORD. 1953. (McGraw-Hill)

1. To those accustomed to the present-day style of geometrical teaching in England this interesting book will appear to be a curious mixture of ancient and modern. The former is represented by the very elaborate treatment of some seventy theorems (in which number the constructions are included); the latter by a variety of additions and connected topics. The theorems are not reached till chapters on Basic Constructions, Fundamental Geometric Constructions and Experimentation have led up to a chapter on the Assumptions which are to be the basis of the subsequent reasoning. Of these there are no fewer than forty-three, eleven axioms such as " halves of equals are equal " and thirty-two geometrical postulates, including " through two given points it is possible to draw one, and only one, straight line ", and the assumptions about angles at a point, angles made with parallel lines and the conditions for congruence of triangles now usual.

It is not the making of these assumptions but the way they are to be used that will strike the reader as old-fashioned. In writing out a proof of one of the theorems the page is to be divided into two parts vertically, the first half headed " Argument " and the other " Defense " (*sic*) and every step in the argument is numbered and has to be defended ; the pupil is likely, I think, to get tired of defending the step " Draw DS ", presumably by referrring to the assumption quoted above. It is not quite clear by what wording this reference is to be made, for in only a few cases are the " defense " steps printed in the book, what is printed being usually " why possible? " or " why? ", though the less obvious answers are given, such as " By case 1 of this theorem ".

This lack of completeness in the final proofs of the theorems is however more than compensated by the elaborate ways in which the pupil is led up to them.

Each theorem is given in three groups ; in group I the enunciations are given and the pupil is asked to develop the proofs ; in group II hints are given to suggest how to develop them, and finally in group III the theorem is given in full with general and particular enunciations, figures and proofs in which every step of the argument is fully given, though the defense is left as stated above.

There are a satisfactory number of exercises on each set of theorems ; thus in the chapter on " Circles and Circle Measurement " which contains Theorems 33 to 41 there are a group of 10 questions, then a group of 62, mainly what we call " riders ", then a group of 24 " numerical exercises " and then 23 headed " Judging relationships " in which one of the words " always ", " sometimes " or " never " has to be inserted into such sentences as

" The longer the chord of a circle, the nearer it ? is to the centre,"

" Two circles can ? have two common chords."

There are many striking and interesting additions to the austerity of geometrical proof. Perhaps the most noteworthy is the vigorous attempt to show how methods of reasoning used in geometry are also applied to other departments of life. In twenty sections, examples are given of correct and incorrect reasoning from correct and incorrect assumptions. A few instances follow :

" All churches are painted white ; this building is painted white ; therefore this building is a church." Same with last two clauses interchanged.

" All boys with red hair are named Tom ; this boy has red hair ; therefore . . ."

" Separate into the ' if and then ' form : ' Traffic is slowed down by a heavy snowfall '."

" What hidden assumption is made in ' Mr. E. who is to direct our school orchestra next year will probably be very temperamental '."

Other features are :

(1) Historical notes set out in a very interesting way ; for example, the parallel postulate question under the heading " Who told the truth ? "
(2) Admirable pictures of " Geometry in life " showing buildings, designs for fancy buttons, sets of initial letters, etc., stated to be from the note-books of named pupils.
(3) " Add a dimension." Seven sections on bits of solid geometry inserted.
(4) " Keep the iron hot." Sets of review exercises in arithmetic and algebra, headed " Set your goal at 100 per cent. accuracy."
(5) Chapter on " Food for thought " (optional). Some analytical geometry and a discussion on the fourth dimension.

Altogether this is a noteworthy book. Unfortunately it is a pupil's rather than a teacher's book and in this country the pupil would have little use for the elaborate proofs of so many theorems ; also the book is expensive. The teacher, however, could well use it for oral work on the non-geometrical reasoning and for many suggestions as to method.

2. This book is on the same lines as the *Plane geometry*. Since it contains some sixty-seven theorems—set out as before with " argument " and " defense "—it departs further than the other book from the present-day practice of this country. The additional matter ; exercises on every-day reasoning ; judging relationships ; those headed " keep the iron hot " as well as riders, are as attractively set out as before. It is interesting to be asked to give the converse, inverse, and contrapositive of the statement " If a person steals a car he is breaking the law ".

The teacher can get many suggestions from this book, but on the whole will find it less useful than the other.

C. O. TUCKEY

Thermodynamics and Statistical Mechanics. By WILLIAM P. ALLIS and MELVIN A. HERLIN. Pp. viii, 239. 51s. 1952. (McGraw-Hill)

The text is divided into two equal parts. The first deals with thermodynamics, the approach to the subject being empirical with the emphasis on the mathematical formulation rather than on experimental detail. The equations of state, the laws of thermodynamics and a few simple applications are about all that can be put into a hundred pages written in an informal style. The mathematical formulation is clear and well done.

The second part deals with statistical mechanics where the informal exposition of the relation between entropy and probability is less satisfactory. Stirling's approximation is used to compute the probabilities and the subject matter is presented in the usual classical form leading to the Maxwell-Boltzmann distribution laws. There are two hurried chapters at the end on the modifications due to quantum theory.

The authors claim the book to be a half-year's lectures to senior physics students at M.I.T. It is certainly no treatise for the specialist nor a suitable text-book for physics degree courses ; but for the physics student who wants a short, neatly produced introduction to the subject in mathematical form it may well be recommended (if price is no object!)

R. L. R.

Statistical Tables and Problems. By ALBERT E. WAUGH. Third edition. Pp. xiv, 242. 25s. 6d. 1952. (McGraw-Hill, New York)

This American book of statistical tables, formulae, and problems has reached a third edition without becoming well-known in this country.

The tables and their accompanying descriptions occupy 91 pages. They consist of a rather incongruous assortment of arithmetical, elementary statistical, and other tables (*e.g.* of the Perpetual Calendar and the Greek alphabet) but little of interest to statisticians that is not more conveniently and cheaply available elsewhere.

The 137 formulae seem to summarise the results derived in the author's *Elements of Statistical Method*. They are quoted here without explanation of their notation and without comment about their applicability.

The "problems" (135 pages), which range in subject from graphical presentation to the computation of correlation coefficients, consist in part of tabulated statistical data drawn from a variety of sources which could form the basis of problems. No answers are given.

The only readers in this country to whom the book is likely to appeal are examiners in elementary statistics who are hard-pressed for time and material.

B. C. BROOKES.

Measure and Integration. By M. E. MUNROE. Pp. x, 310. $7.50. 1953. (Addison-Wesley, Cambridge, Mass.)

This is a most enjoyable book : strictly modern in its "postulational" build up, rigorous but not unduly abstract, leading the reader well up to recent results and problems in the theory of measure and integral. I like the vigorous colloquial American style in which it is written ; it is never dull nor tiring. The author has intended his book for post-graduate courses, and it is excellently suited for this purpose. It presupposes, I think, some previous acquaintance with the classical concepts connected with, say, the names of Riemann, Stieltjes, and Lebesgue. They appear, of course, at their proper places in the general theory, but it is, in my opinion, quite generally true that one ought to know something about particular important cases of an abstract theory, as a sort of concrete background, before one can appreciate that

theory properly. In fact, abstract postulational theory usually follows the concrete case by analysis of its concepts and arguments, and does not precede it as it appears to do in its final form. This is particularly true in the present case. The reader will find that the classical case of the Lebesgue-Stieltjes measure and integral does not only have all the relevant features of the general situation, in its most satisfactory form, but that it is still the most important of all cases.

Of the rich content of the book we give the following indications. Chapter I sets the general background of point set theory, metric and linear spaces, and of additive classes of sets and Borel sets. In Chapter II the general theory of measure in abstract space is developed. This is done, following Carathéodory, by first introducing sub-additive outer measures and then characterising the measurable sets by a suitable property, so that for them the outer measure becomes completely additive. In Chapter III, special measures are discussed : the n-dimensional Lebesgue-Stieltjes measure, probability measures, Hausdorff and Haar measures. Chapter IV deals with measurable functions and their limiting properties. In Chapter V the theory of integration begins. The integral is first defined for " simple " functions, *i.e.* linear combinations of characteristic functions of disjoint measurable sets : a certain notion of measure is fixed in advance. The integral of a general measurable function is then obtained by approximation through simple functions. Apart from the usual properties of the integral found in the common textbook literature, we find here the Hahn-Decomposition theorem, the theorem of Radon-Nikodym and that of Fubini. The n-dimensional Lebesgue-Stieltjes integral and distribution integrals in the theory of probability illustrate this chapter. Chapter VI contains a thorough discussion of the different notions of convergence in abstract spaces ; in particular, convergence in measure and mean convergence. The spaces L^p and their functionals are treated, and orthogonal expansions in Hilbert space. A very readable proof of the mean ergodic theorem is also given. The final Chapter VII contains Vitali's covering theorem and its application to the delicate problems concerning the relation between differentiation and integration. This is done here for the n-dimensional Lebesgue-Stieltjes integral. Ample and very illuminating exercises supplement the account. The book is beautifully printed. W. W. ROGOSINSKI

Inhalt und Mass. By K. MAYRHOFER. Pp. viii, 269. Paper 61s. 6d. ; linen 66s. 6d. 1952. (Springer, Vienna)

This book on " content and measure " is very much a book for the specialist in the axiomatic analysis of the ideas underlying the notion of an abstract measure. Prof. Mayrhofer who himself has made important contributions in this field of research has, no doubt, given a very competent and reliable account of all this. It seems, however, to the reviewer that the treatment is rather too elaborate, if not old-fashioned. After all this painstaking analysis no other examples are discussed than the familiar ones of the Peano-Jordan content and the measures of Borel and Lebesgue : the most important Lebesgue-Stieltjes measure is not even mentioned. Nor is the notion of the integral, so intimately related to that of measure, touched upon. Instead much space is taken up by discussing volumes of certain elementary sets in n-dimensional space. A final chapter on measure in Boolean structures may be of novel interest to the specialist. The book is well printed. W. W. ROGOSINSKI

Conformal Mapping. By L. BIEBERBACH. Translated by F. Steinhardt. Pp. vi, 234. $2.25. 1953. (Chelsea Co., New York)

In translating the latest German edition of Bieberbach's " Einführung in die Konforme Abbildung " the translator has been able to convey in English

much of the lucidity and precision of the German original. This is the first book in English to give an elementary, readable account of the Riemann Mapping Theorem and the distortion theorems and uniformisation problem, with which it is connected.

The book is in five sections. The first two which deal with Foundations, Linear Functions, and Rational Functions are introductory. The third section contains two important topological principles. The fourth and fifth sections of this book which discuss a number of mappings in detail are characteristic of the author and make the book extremely useful and valuable to the complex variable analyst. Finally there is a short bibliography and an index.

Nearly all the material in this book is classical but it is presented here in a very readable and attractive form. There is no reason why it should not be read by undergraduates although it goes much further than is necessary for an undergraduate course. H. G. EGGLESTON

Geometrie der Zahlen. By HERMANN MINKOWSKI. Pp. 256. $4.50. 1953. (Chelsea Publishing Company, New York)

Many will be glad that this classic has been reprinted. It deals not only with the geometry of numbers as now understood (mainly for convex bodies) but also with the general theory of convex bodies. Both subjects owe a great deal to Minkowski and both have been greatly developed subsequently. There is a comprehensive report of Bonnesen and Fenchel (1933) dealing with convex body theory; but this book (1896) and Minkowski's *Diophantische Approximationen* (1907) are still, apparently, the only general expositions of the geometry of numbers in book form, apart from Hancock's peculiar compilation. J. W. S. C.

Electric Circuit Theory and Operational Calculus. By J. R. CARSON. 2nd edition. Pp. vii, 197. $3.95. 1953. (Chelsea Publishing Company, New York)

This book is a faithful reprint of a well-known work originally published in 1926.

J. R. Carson was one of the first to attempt to place Operational Calculus on a rigorous mathematical basis, and his approach was to derive Heaviside's operational formulae by means of an integral equation. This, perhaps, is not the conventional approach of today, but the Carson (Heaviside) solution and corresponding operational formula are identical with a function and its Laplace Transform, if the latter is defined as

$$\int_0^\infty pe^{-pt} f(t)\, dt.$$

To illustrate the theory, a comprehensive discussion of the electrical line equation is included, and the book could be highly recommended on this account alone. "Carson" should be in the library of every University and Technical College; if the work of some of the latter does not go to this standard, the book should nevertheless be there for the staff and as a possible inspiration to the abler student. M. HUTTON

Mathematical Recreations. By MAURICE KRAITCHIK. Pp. 330. $1.60 paperbound, $3.00 clothbound. (Dover Publications, New York)

This is the second revised edition of the book originally published in England in 1943, and is one of a series which the publishers are making available in paper covers as well as in normal bindings. The paperbound copy costs little

more than half the clothbound edition, so the practice is one which students might welcome at a time when books are so dear. There are very few alterations, and the book appears to be a photographic reprint of the earlier edition. Some errors have been corrected, the information on Mersenne numbers has been brought up to date, and the only new material is a two-page analysis of the False Coin Problem which seems to have been hastily prepared, and gives the number of coins which can be sorted in n weighings wrongly.

The book can be strongly recommended to all who are interested in the lighter side of mathematics. It includes puzzles of all grades of difficulty, a full account of magic squares, a good section on mosaics, the theory of a large number of games of which many will be new to most readers, and chapters on the calendar, probabilities, and problems of the Chess board.

Any mathematician looking for a bedside book, or a teacher looking for something different for the end of term will find it a rich store of treasure, and every school library should have a copy on the shelf beside Rouse Ball's book with a similar title.

T. J. F.

Lectures on Cauchy's Problem in Linear Partial Differential Equations. By J. HADAMARD. Pp. v, 316. $1.70; cloth $3.50. 1952. (Dover Co., New York)

The now classical 1923 edition of this book has been reissued without textual alterations. Written as a research monograph, the chief value of the book is now as an introduction to the subject and from this point of view it is still a readable and stimulating account of linear partial differential equations. But it is inevitable that it should be of only limited value to the research worker.

H. G. EGGLESTON

Elements of the Theory of Functions. By K. KNOPP. Pp. 140. $1.25; cloth $2.25. 1952. (Dover Co., New York)

This booklet is translated from the well-known German version and is the introductory volume of a five-volume set of short books by Knopp that together form a complete undergraduate course on the Theory of Functions.

The translation follows the original closely; sometimes it does so too closely as the following sentence from page 15 shows,

"This interpretation urges one, as it were, to attempt, with the new impossibilities discussed in §4 in mind, to define a set of operations for the points and vectors of the plane (see §3), and in this way create a system of elements to which the deficiencies of the system of real numbers no longer adhere."

In fairness to the translater it should be stated that such examples are rare and the booklet has the rigour and thoroughness that were present in the German Edition.

H. G. EGGLESTON

Éléments de Mathématique XIV. Algèbre: Groupes et corps ordonnés; Modules sur les anneaux principaux. By N. BOURBAKI. Pp. 161. 1800 fr. 1952. (Hermann, Paris)

This volume of Bourbaki's well-known treatise consists of two chapters, one on ordered groups and fields, and the other on modules over principal ideal rings. The first chapter consists largely of definitions and of fairly immediate deductions from them. The groups considered are assumed to be abelian, and particular attention is paid to lattice ordered groups. The author has in mind throughout the application of the theory that will be made in the second chapter, when the group will be the group of fractional ideals of the quotient field of a principal ideal ring, and the order relation that of

divisibility ; and he translates the relevant results into the language appropriate to this application as he goes along. The ordered fields considered are totally ordered, and the main theorems are those on maximal ordered fields. All this is applied in the second chapter to the theory of modules over a commutative principal ideal ring, which now runs very sweetly indeed. The unique factorisation theorem is first proved, and applied to the special cases of rational integers and polynomials in one variable over a field. Then follows the decomposition of a module with no free submodule into primary components ; and the basis theorem for finitely generated modules. Here the applications are to the basis theorem for abelian groups, and to the normal form theorem for linear transformations of a vector space into itself.

Anyone who is likely to be interested in this volume is presumably already aware of the virtues and defects of the treatise as a whole, which makes the reviewer's task lighter ; but it should be said at once that in this volume the virtues shine particularly brightly, and the defects are scarcely present at all. This is, I think, because the abstract approach in this case pays really handsome dividends, and because the subject matter is such that the volume can be read with understanding without a continuous effort to bear in mind the oddities of Bourbaki's terminology. The exercises here are also particularly rich. Besides those designed to ease the reader's passage from the abstract theory to the very concrete applications, there are others which assist the reader to such diverse results as the first approximation to the prime number theorem, the determination of the quadratic fields with Euclidean algorithm (excluding, of course, the difficult case of real fields with $d \equiv 1$ (4)), and the theorem of Ulm-Zippin on countable primary abelian groups. Altogether this is a book for which the algebraist should be most grateful. G. H.

Bessel functions and formulæ. Compiled by W. G. BICKLEY. Pp. xxx–xl. 3s. 6d. 1953. (Published for the Royal Society at the Cambridge University Press)

The last of the series of British Association Mathematical Tables was Vol. X : Bessel functions, Part II. This extract from the introduction contains a summary of notations, and a most useful and detailed list of important formulae, in a readily accessible and beautifully printed form. The compact pamphlet should be an acceptable aid to all users of Bessel functions.

T. A. A. B.

WANTED

H. R. HAMLEY : *Relational and Functional Thinking in Mathematics* (Ninth Yearbook of the National Council of Teachers of Mathematics).

Would any member willing to sell this book please communicate with A. G. Sillitto, Training College, Jordanhill, Glasgow, W.3.